ON THE REGULARITY DIMENSIONS OF MEASURES

Douglas Charles Howroyd

A Thesis Submitted for the Degree of PhD
at the
University of St Andrews

2020

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Douglas Charles Howroyd

University of
St Andrews

This thesis is submitted in partial fulfilment for the degree of

Doctor of Philosophy (PhD)

at the University of St Andrews

March 2020

Acknowledgements

There is not enough space in this margin to thank everyone who deserves to be thanked for being part of my PhD journey. It has been an amazing 4 years and I have had the pleasure of having many great friends during this time. In particular, I would like to start by thanking my friend and supervisor Jonathan Fraser who has been supporting me since my Master's project in Manchester all those years ago. Without his extensive knowledge and time, none of this thesis would be possible. But he has also taught me much about non-mathematical topics and, for that, I will always be grateful. I could not have asked for a better supervisor. I would also like to thank Kenneth Falconer for uncountable interesting mathematical conversations and for providing much feedback on several papers and this thesis. I would also like to thank the EPSRC for funding me with the Doctoral Training Grant (EP/N509759/1).

Like the many analysis PhD students from St Andrews that have come before me, I would like to thank all the inhabitants of the bunker, this includes Amlan, Ewa, Han, Lawrence, Liam, Sascha, Stuart and Vuksan. Along with the rest of the analysis group - Mike, Lars, Collin, Tom Kempton and Jonathan Hickman - lunches and Whey Pat trips were always bright and fun times, sometimes even educational!

For 2 years I had the pleasure of sharing Muckross Flat 4 with a number of individuals and thoroughly loved that time. I would like to thank all of these flatmates, with a special place in my heart reserved for Ashley and Felix.

Over the past few years I have met many different mathematicians and learnt much from them. Our collaboration with Antti was a particularly enjoyable experience, despite some amount of stress at the end. The Manchester analysis and dynamics group have always been my second home, welcoming me in whenever I was around. Specifically Charles who has been putting up with me for 8 years now.

Finally I would like to thank my family and friends for being behind me every step of the way, especially Sean and Amber. To anyone else reading this, I apologize for missing you out, you're breathtaking.

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Abstract

This body of work is based upon the following three papers that the author wrote during his PhD with Jonathan Fraser and Han Yu: [FH20, HY17, How19].

Chapter 1 starts by introducing many of the common tools and notation that will be used throughout this thesis. This will cover the main notions of dimensions discussed from both the set and the measure perspectives. An emphasis will be placed on their relationships where possible. This will provide a solid base upon which to expand. Many of the standard results in this part can be found in fractal geometry textbooks such as [Fal03, Mat95] if further reading was desired.

The first results discussed in Chapter 2 will cover some of the regularity dimensions' properties such as general bounds in relation to the Assouad and lower dimensions, local dimensions and the L^q-spectrum. The Assoaud and lower dimensions are known to interact pleasantly with weak tangents and these ideas are discussed in the regularity dimension setting. We then calculate the regularity dimensions for several specific example measures such as self-similar and self-affine measures which provides an opportunity to discuss the sharpness of the previously obtained bounds. This work originates in [FH20] where the upper regularity dimension was studied, with many of the lower regularity dimension results being natural extensions.

In Chapter 3 we continue the study of the upper and lower regularity dimensions with an emphasis on how they can be used to quantify doubling and uniform perfectness of measures. This starts with an explicit relation between the upper regularity dimension and the doubling constants along with a similar link between the lower regularity dimension and the constants of uniform perfectness. We then turn our attention to a technical result which can be made more quantitative thanks to the regularity dimensions. It is interesting to study how properties, such as doubling, change under pushforwards by different types of maps, here we study the regularity dimensions of pushforward measures with respect to quasisymmetric homeomorphisms. We round this chapter out with an interesting application of the lower regularity to Diophantine approximation by noting the equivalence between uniform perfectness and weakly absolutely α-decaying measures. The original material for this part can be found in [How19] with part of the first section integrating a result of [FH20].

Finally, in Chapter 4, we will consider graphs of Brownian motion, and more generally, graphs of Lévy processes. This will involve the calculation of the lower and Assouad dimensions for such sets and then the regularity dimensions of measures pushed onto these graphs from the real line. These graphs are the only examples in this thesis for which the Assouad and lower dimensions had not been previously calculated so we delve deeper into the area, studying graphs of functions defined as stochastic integrals as well. This chapter is based on the paper [HY17] for the set theoretic half, with the regularity dimension results coming from [How19].

Contents

Chapter 1

Measures and dimensions

Felix Hausdorff introduced what is now known as the Hausdorff dimension in 1918 [Hau18] as a means of studying geometric objects for which the classical notions of dimension and size fail. The sets studied in this context are often irregular, such as trajectories of Brownian motion, the Cantor set and the Mandelbrot set; see Figures 1 and 2 for a few examples. Whilst there is no formal definition behind the term 'irregular objects', one should think of them as possessing infinitesimal detail, that is, as one zooms in on a certain part of the object, new, interesting behaviours are exhibited. Mandelbrot [Man77] coined the term *fractal* in 1977 to encompass the wide range of examples satisfying this concept and since then there has been an increasing interest in the study of fractals, both theoretical and applied. As many naturally occurring objects are messy and chaotic, fractals are suitable models for their study, seeing use in a wide range of disciplines such as finance and the life sciences, as in [DI16, PMR$^+$19]. The applications in theoretical mathematics are also numerous, having profound influences on number theory and dynamical systems among others.

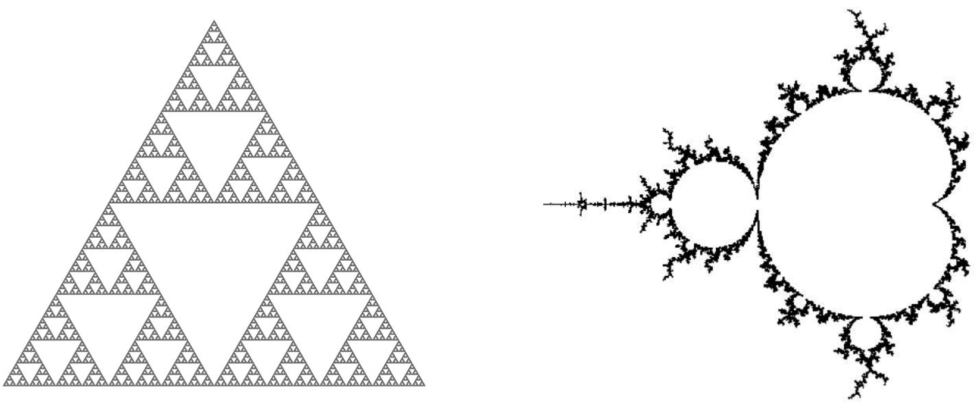


Figure 1: A few well studied fractal sets. The first one is a Sierpiński triangle and the right one is the boundary of the Mandelbrot set.

Since the introduction of the Hausdorff dimension, many other notions of dimension have been defined, each with its own unique properties. The underlying principle of all definitions of dimension is that they should quantify how much of a set is within a given space asymptotically; how they quantify this and which space they restrict to is what distinguishes them. We will mostly be interested in the *Assouad* and *lower dimensions* in this thesis, but one of the interesting aspects of fractal geometry is the interaction between the different notions of dimension, and so other dimensions will arise throughout to place this work into the greater context. The Assouad dimension, when thinking about the governing idea of a dimension, is interested in understanding how much space a given set takes up with respect to a small scale r when restricted to a ball of size R, where R is close to zero but still greater than r. In particular, it focuses on the balls in which the studied set takes up as much space as possible. This means the Assouad dimension provides an *extremal* notion of size of an object, often being the largest of the main dimensions, and discards information about the rest of the set. This is complemented by the lower dimension, which provides similar details but focusing on where the set is the least space filling.

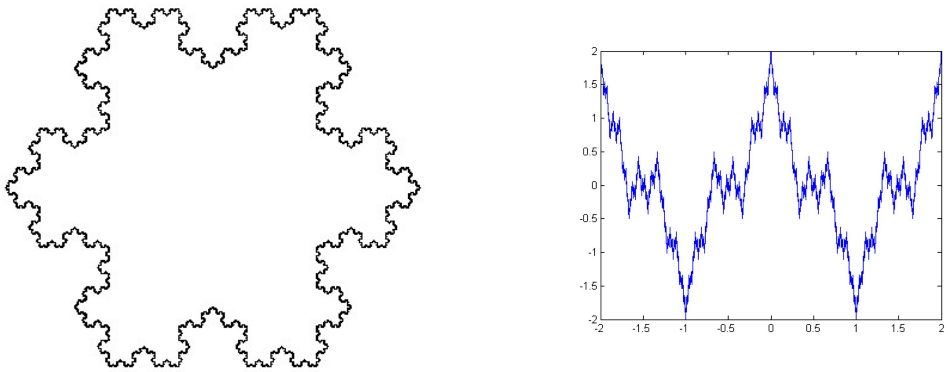


Figure 2: More examples of fractals. Left is a von Koch snowflake and right the graph of the Weierstrass function.

During the study of these irregular geometric objects, a desire to study the regularity of measures supported on them has arisen, leading to the notion of the dimension of a measure. Naturally there are several definitions of dimension of a measure, each one often being related to a dimension of sets. The study of these dimensions of measures has recently contributed to significant advances in our understanding of sets that were previously considered almost too complex to work with, as in [Hoc10]. If a geometric dimension tells us roughly how much of a set is within a given area, a dimension of measures is interested in how the measure is distributed over such objects. The Assouad and lower dimensions have corresponding dimensions of measures which, in this thesis, will be called the *upper* and *lower regularity dimensions* respectively. Whilst the Assouad and lower dimensions have seen an increasing popularity in the recent years, the regularity dimensions are only starting to attract attention and much of this thesis will be interested in expanding on their basic properties and their interactions with other dimensions. Like the Assouad dimension, the regularity dimensions are also extremal notions of size, a fact exemplified by noting that the upper regularity dimension of non-doubling measures is always infinity. The links between doubling measures, a standard regularity property in the literature, and the upper regularity dimension will be of particular interest in this thesis.

1.1 Dimensions of sets

In this section we will introduce and define several different notions of dimension that will occur throughout this thesis. Dimensions are concerned with the size of objects and so one must first establish a notion of distance. This is done by studying subsets of metric spaces (X, d); for clarity we will often refer to the space X with the metric being implicitly defined. One should assume X is compact, however this is not always needed. In fact there are some particularly beautiful results that will be referenced in Section 1.1.2 that hold for non-compact sets, which are helpful when studying number theoretic questions. When discussing the distance between points we will use the notation $d(x, y)$ to be the distance between the points $x, y \in X$ where the space is clear. If multiple metric spaces are being considered simultaneously the notation $d_X(x, y)$ will be adopted to specify the metric space X in which the distance is taken. We will later study measures on metric spaces and we take the associated σ-algebra to be the Borel σ-algebra given the underlying metric. We will not constantly specify the σ-algebra for notational convenience.

Whilst we aim to be precise here, this will still be a brief introduction and the interested reader should see the standard books [Fal03, Fal97, Mat95] for an extensive introduction to many of these topics.

1.1.1 Hausdorff and box dimensions

The Hausdorff dimension is arguably the most commonly studied notion of dimension in fractal geometry literature and has been since its inception in 1918 by Hausdorff. Whilst its construction is somewhat technical, involving the Hausdorff measure, the advantages are significant, endowing this dimension with many interesting properties.

Let $F \subseteq X$, $s \geqslant 0$ and $r > 0$. We first define a r-cover of F as a cover of F by open sets of diameter at most r. We will use $|\cdot|$ to denote the diameter of a set, so $|F| = \sup\{d(x,y)\colon x, y \in F\}$ for any $F \subseteq X$. Then the r-approximate s-dimensional Hausdorff measure of F is defined by

$$\mathcal{H}^s_r(F) = \inf\left\{\sum_{i\in I}|U_i|^s\colon \ \{U_i\}_{i\in I} \text{ is a } r\text{-cover of } F\right\}.$$

Taking the limit as $r \to 0$ gives the s-dimensional Hausdorff (outer) measure $\mathcal{H}^s(F) = \lim_{r\to 0}\mathcal{H}^s_r(F)$. This finally leads to the *Hausdorff dimension* of F given by

$$\dim_{\mathrm{H}} F = \sup\{s \geqslant 0\colon \mathcal{H}^s(F) = \infty\} = \inf\{s \geqslant 0\colon \mathcal{H}^s(F) = 0\}.$$

As mentioned before, calculating this dimension from first principles is usually quite difficult, but due to the definition following from the Hausdorff measure it is well behaved. In particular, it is countably stable. We will later be interested in calculating the Hausdorff dimension of a few different sets, notably self-similar sets with the Strong Separation Condition (SSC), Bedford-McMullen carpets and sequences of numbers decaying to zero, such as $\{1/n\colon n \in \mathbb{N}\} \cup \{0\}$. For the first two, a formula already exists which will be stated when needed and for the last one, as it is a countable set, it is easily seen to have zero Hausdorff dimension due to the countable stability of the dimension.

A coarser but easier to compute dimension was desired, without straying too far from the Hausdorff dimension. This led to the upper and lower box dimensions in the 1920s, as in [Bou28]. Given $F \subseteq X$, take $N(F, r)$ to be the smallest number of sets needed in a r-cover of F. Then the *upper* and *lower box dimensions* are defined by

$$\overline{\dim}_{\mathrm{B}} F = \limsup_{r\to 0}\frac{-\log N(F,r)}{\log r} \quad \text{and} \quad \underline{\dim}_{\mathrm{B}} F = \liminf_{r\to 0}\frac{-\log N(F,r)}{\log r}.$$

When $\overline{\dim}_{\mathrm{B}} F = \underline{\dim}_{\mathrm{B}} F$, we simply talk about the *box dimension* of F and denote it by $\dim_{\mathrm{B}} F$. This will be the case in all the examples in this thesis for which the box dimension is studied. In reality we will take $N(F, r)$ to be the smallest number of balls of radius r required to cover F, or the smallest number of squares of side lengths r. These changes will not modify the box dimensions of a set and will greatly simplify calculations. Of particular interest in real world applications, one can take $N(F, r)$ to be the number of squares in the r-mesh that intersect F and a computer can then calculate this number for a sequence of scales, approximating the box dimension to a reasonable degree, such as in [DI16].

A simple example of the box dimension's coarser behaviour is the set $F = \{1/n \colon n \in \mathbb{N}\} \cup \{0\}$ which has box dimension $1/2$ despite having zero Hausdorff dimension. The proof of this is not trivial so we include it here for completeness.

Let $r > 0$, we wish to study $N(F, r)$ for r small. First note, that for any $k \in \mathbb{N}$,

$$\frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)}.$$

Thus the distances between points in this set behave like $1/k^2$. Choose $n \in \mathbb{N}$ to be the largest integer such that $1/n(n+1) > r$. We will then split the cover of F into two parts, one half where the distances between points are smaller than our covering sets and the other where they are greater. In the second case it is clear that one ball is needed for each element for a full cover, so n balls. When the distances between points are smaller than r we will need an essentially maximal covering to cover all such elements as the set starts to behave like an interval with respect to the scale r. Therefore covering the first section is akin to covering the interval $[0, 1/n]$. Hence

$$N(F, r) \geqslant N([0, 1/n] \cap F, r) \geqslant n/2 \geqslant r^{-1/2}/4$$

and

$$N(F, r) \leqslant N([0, 1/n], r) + n \leqslant \frac{3}{4} r^{-1/2},$$

completing the proof that $\dim_{\mathrm{B}} F = 1/2$.

1.1.2 Assouad and lower dimensions

The notion of dimension that will most interest us in this thesis is the Assouad dimension, named after Patrice Assouad who proved the Assouad embedding theorem in the 70s [Ass77, Ass79] which relates the Assouad dimension of a metric space with nearly bi-Lipschitz embeddings into Euclidean space. The notion of the dimension itself can be traced back to a paper of Bouligand in 1928 [Bou28] and then Larman in the 60s [Lar67a, Lar67b]. The *Assouad dimension* of $F \subseteq X$ is defined by

$$\dim_{\mathrm{A}} F = \inf \Bigg\{ s \geqslant 0 \ : \exists \text{ constant } C > 0 \text{ such that, for all } \ 0 < r < R,$$
$$\text{we have } \sup_{x \in F} N_r(B(x, R) \cap F) \leqslant C \left(\frac{R}{r}\right)^s \Bigg\}.$$

Here and throughout, $B(x, R)$ denotes a closed ball in X of centre x and radius R.

As the supremum is taken over all points in the set F, the dimension returns local information, specifying how the set behaves in its most dense regions. This contrasts with the global averages that the Hausdorff and box dimensions provide. Usually the Assouad dimension will be the largest of dimensions a space can take. This local 'largeness' is important in embedding problems since if there is an embedding that holds for the most extreme subsets, then it will likely hold for the rest of the set.

Whilst the name comes from an application to embedding theory, the Assouad dimension has seen much study in a number of different areas, including further embedding theory questions [Ols02, OR10, Rob10], quasi-conformal geometry [Hei01, Tys01, MT10], classic fractal geometry [FK18, FT18, Tro17, Orp19, LDR13] and even some applications to number theory [FY16, FSY17]. This list is non-exhaustive and we recommend [Rob10, Fra14] for further discussion of this dimension including many of its basic properties, however these do not cover the multitude of new results which have appeared in the past few years.

A key notion that is highly related to the Assouad dimension is the concept of a doubling space. A space X is doubling if there exists a constant $M > 0$ such that, for any $R > 0$ and any $x \in X$, one can cover the ball of radius R and centre x by at most M balls of radius $R/2$. It is known that a space is doubling if and only if it has finite Assouad dimension, see [Rob10, Lemma 9.4] for a proof. Thus, the Assouad dimension can be seen as quantifying how doubling a space is.

We are also interested in the following measure theoretic formulation of the Assouad dimension due to [LS98, KV88] for closed subsets F of complete metric spaces X

$$\dim_{\mathrm{A}} F = \inf \Bigg\{ s \geqslant 0 \ : \exists \text{ a Borel probability measure } \mu \text{ fully supported by } F \\ \text{and constant } C > 0 \text{ such that, for all } \ 0 < r < R, \\ \text{we have } \sup_{x \in F} \frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant C \left(\frac{R}{r}\right)^s \Bigg\}.$$

As the spaces X are assumed to be compact, the completeness condition is always satisfied for the examples studied here. This definition allows us to relate the purely geometric concept of dimension with a measure theoretic study. This can be advantageous when more is known about the measures than the set itself, such as in [FH15]. More importantly, this will lead to the notion of 'Assouad dimension' of a measure, which will be further explored in the next section.

As the Assouad dimension provides coarse, extremal information, it is natural to consider an analogous dimension which studies regions of the set which require very few balls to cover. This leads to the definition of the lower dimension, first studied by Larman [Lar67a, Lar67b]. The *lower dimension* of $F \subseteq X$ is

$$\dim_{\mathrm{L}} F = \sup \Bigg\{ s \geqslant 0 \ : \exists \text{ constant } C > 0 \text{ such that, for all } \ 0 < r < R \leqslant |F|, \\ \text{we have } \inf_{x \in F} N_r(B(x,R) \cap F) \geqslant C \left(\frac{R}{r}\right)^s \Bigg\}.$$

A careful reader would remark that this definition requires the radii $r < R$ to be bounded above by the diameter of F. This is not needed in the Assouad dimension but here, without this condition, the lower dimension would always be zero for bounded sets. As the objects studied in this thesis are compact this requirement is easily satisfied. For unbounded sets one simply takes the diameter to be infinite and the same definition holds. The lower dimension has seen less study, partially due to a few undesirable properties that it possesses, but when studying many of the main examples of fractals it behaves as expected and results for the Assouad dimension can often be transposed to the lower dimension.

Like the Assouad dimension, the lower dimension also has a related regularity property. A space X is uniformly perfect if there exists a constant $K > 1$ such that for any $x \in X$ and $R > 0$, there exists $y \in X \cap (B(x,R) \setminus B(x,R/K))$. One can in fact show that a set is uniformly perfect if and only if it has positive lower dimension, see [KLV13, Lemma 2.1] where the lower dimension is called the lower Assouad dimension. One of the main problems with the lower dimension is that the lower dimension of any set which contains an isolated point is always zero. Thus, restricting to uniformly perfect spaces would easily avoid this difficulty.

Bylund and Gudayol [BG98] proved the following result linking measures and lower dimension, similar to [LS98, KV88]. Let F be a closed subset of the complete metric space X, then

$$\dim_{\mathrm{L}} F = \sup \Bigg\{ s \geqslant 0 \ : \exists \text{ a doubling Borel probability measure } \mu \text{ fully} \\ \text{supported on } F \text{ and constant } C > 0 \text{ such that,} \\ \text{for all } \ 0 < r < R \leqslant |F|, \text{ we have } \inf_{x \in F} \frac{\mu(B(x,R))}{\mu(B(x,r))} \geqslant C \left(\frac{R}{r}\right)^s \Bigg\}.$$

Compared to the measure theoretic definition of the Assouad dimension, this definition is essentially as expected, except for the restriction to doubling measures.

A measure μ supported on X is doubling if there exists a constant $C(2) > 1$ such that for all x in the support of μ and $R > 0$ we have $\mu(B(x,R)) \leqslant C(2)\mu(B(x,R/2))$; these measures will be discussed further in the next section. Restricting to doubling measures forces the measure to truly capture the entirety of the set, in the same way that focusing on fully supported measures is required to avoid part of the space being completely ignored and therefore losing vital information. However, this is not needed for the Assouad dimension as non-doubling measures will provide infinity as the upper bound to the dimension and so will not affect the Assouad dimension.

For a general compact set F the following relations can be found in [Fal03, Lar67a, Lar67b]:

$$\dim_{\mathrm{L}} F \leqslant \dim_{\mathrm{H}} F \leqslant \underline{\dim}_{\mathrm{B}} F \leqslant \overline{\dim}_{\mathrm{B}} F \leqslant \dim_{\mathrm{A}} F.$$

These inequalities can be, and often are, strict, with many interesting examples existing in the literature demonstrating the different combinations of equalities and inequalities possible. For example we have seen that the set $F = \{1/n \colon n \in \mathbb{N}\} \cup \{0\}$ has zero Hausdorff dimension and box dimension equal to $1/2$. As the lower dimension is a lower bound to the Hausdorff dimension, F will also have lower dimension zero and we will prove shortly that it has full Assouad dimension. On the other hand, for a self-similar set satisfying a suitable separation condition, all of these dimensions are equal.

The proof that $F = \{1/n \colon n \in \mathbb{N}\} \cup \{0\}$ has full Assouad dimension is quite similar to the proof of its box dimension, we show this to get a better feel for the Assouad dimension.

As we wish to prove the Assouad dimension is full, we simply need to find a lower bound as the upper bound follows from the dimension of the ambient space. The most common way of doing this for the Assouad dimension is to find sequences of $x_n \in F$, and $0 < r_n < R_n$ such that $R_n/r_n \to \infty$ and $N(F \cap B(x_n, R_n), r_n) \geqslant CR_n/r_n$ for some constant $C > 0$. The extremal behaviour of this set exhibits around 0 so we simply choose $x_n = 0$ for all n. To use the previous work on the box dimension, let $R_n = 1/n$ and $r_n = 1/n^2$. From before, $F \cap B(0, R_n)$ is the part of F where the differences between points is less than $1/n^2$ so covering this subset is like covering $[0, 1/n]$. Thus

$$N(F \cap B(0, 1/n), 1/n^2) \geqslant n/2 = \frac{1}{2}\left(\frac{R_n}{r_n}\right)$$

as desired.

1.2 Dimensions of measures

1.2.1 Upper regularity dimension

As alluded to previously, the measure theoretic definitions of dimension lead to a natural notion of 'Assouad dimension' of a measure which gives a quantifiable description of the extremal local scaling of the measure. This was first defined in [KLV13, KL17] under the name upper regularity dimension and studied in the context of inhomogeneous self-similar sets and Whitney covers. Let μ be a locally finite Borel measure on X and write

$$\operatorname{supp}(\mu) = \{x \in X \colon \mu(B(x,r)) > 0 \text{ for all } r > 0\}$$

for the support of μ. In general, one should assume a measure is locally finite Borel unless otherwise stated. The *upper regularity dimension* of μ is defined by

$$\overline{\dim}_{\mathrm{reg}}\mu = \inf\Bigg\{ s \geqslant 0 \ : \ \text{there exists a constant } C > 0 \text{ such that,} \\ \text{for all } 0 < r < R \text{ and all } x \in \operatorname{supp}(\mu), \\ \text{we have } \frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant C\left(\frac{R}{r}\right)^s \Bigg\},$$

taking $\inf\emptyset = +\infty$.

Combining this definition and the measure theoretic formulation of the Assouad dimension, the following can be deduced for a complete metric space F

$$\dim_{\mathrm{A}} F = \inf\left\{\overline{\dim}_{\mathrm{reg}}\mu \ : \ \mu \text{ is a measure fully supported on } F\right\}. \tag{1.1}$$

This relationship has led to a number of people calling the upper regularity dimension the Assouad dimension, such as in [HHT18, HT19, FK18]. As the papers this thesis is based upon use the terminology upper regularity dimension, we will continue to do the same; however, the term Assouad dimension is growing in popularity and will likely become the standard notation in the future.

As can be seen in equation (1.1), the upper regularity dimension of a measure is always bounded below by the Assouad dimension of its support. In some cases the upper regularity dimension will even attain the Assouad dimension of the space, for example [FH15, Theorem 2.3]. However Käenmäki and Lehrbäck constructed a wide class of sets for which the Assoaud dimension of such a set F is always strictly smaller than the upper regularity dimension of any doubling measure supported on F, see [KL17] and [KV88] for their inspiration. Much of this work will be dedicated to placing the upper regularity dimension in context with other notions of dimension, both generally and for specific examples.

A measure, like a space, can be doubling. Whilst the name doubling comes from the standard definition where measures of two balls are compared, with the radius of one ball being twice that of the other, one can generalise this to any ratio between the radii. Formally, a measure μ is *doubling* if and only if, for some $\theta > 1$, there exists a constant $C(\theta) > 1$ such that for all x in the support of μ and $R > 0$ the following holds

$$\mu(B(x, R)) \leqslant C(\theta)\mu(B(x, R/\theta)). \tag{1.2}$$

It is worth noting here that for a given θ one can choose $C(\theta)$ to be optimal in the sense that

$$C(\theta) = \sup\left\{\frac{\mu(B(x, R))}{\mu(B(x, R/\theta))} \ : \ x \in \operatorname{supp}(\mu), R > 0\right\}.$$

In general, when working with doubling constants, it can be assumed that the constant is always chosen to be as large as possible, we will nevertheless specify this when used.

Recalling a space is doubling if and only if it has finite Assouad dimension, it turns out that the upper regularity dimension behaves similarly. That is a measure is doubling if and only if the measure has finite upper regularity dimension. A proof of this can be found in [JJK$^+$10, Lemma 3.2] and will be further explored in Chapter 3.

1.2.2 Lower regularity dimension

As the lower dimension is a natural dual of the Assouad dimension, so is there an analogue of the upper regularity dimension, called the lower regularity dimension. Inspired by [BG98], a measure μ on a metric space X has *lower regularity dimension* defined by

$$\underline{\dim}_{\mathrm{reg}}\mu = \sup\Bigg\{ s \geqslant 0 \ : \ \text{there exists a constant } C > 0 \text{ such that, for all } 0 < r < R < |\operatorname{supp}(\mu)| \text{ and all } x \in \operatorname{supp}(\mu), \text{ we have } \frac{\mu(B(x, R))}{\mu(B(x, r))} \geqslant C\left(\frac{R}{r}\right)^s \Bigg\}.$$

The lower regularity dimension quantifies the minimal local scaling of a measure and many of the ideas that have been discussed regarding the upper regularity dimension carry over to the lower analogue. As with the lower dimension, R is bounded above by a constant in this definition; without such a constraint the lower regularity dimension would be always zero.

A definition of the lower dimension of a closed set $F \subseteq X$ can be deduced purely in terms of the dimension of measures

$$\dim_{\mathrm{L}} F = \sup \left\{ \underline{\dim}_{\mathrm{reg}} \mu : \ \mu \text{ a doubling probability measure fully supported on } F \right\}. \tag{1.3}$$

This has lead to the lower regularity dimension being called the lower (Assouad) dimension in the literature, such as in [HT19]. This also implies that the lower regularity dimension is a lower bound to the lower dimension, with examples for both equality and strict inequality of the two existing.

Being consistent with the previous section, one can state a definition for a measure to be *uniformly perfect*; such measures were recently called reverse-doubling in [KL17] due to the resemblance with doubling measures. A measure μ is uniformly perfect if for any $\theta > 1$ there exists a constant $K(\theta) > 1$ such that

$$\mu(B(x, R)) \geqslant K(\theta)\mu(B(x, R/\theta)), \tag{1.4}$$

for all x in the support of μ and $R > 0$. Again, for a given θ, one can choose $K(\theta)$ to be as small as possible by letting $K(\theta)$ be the following

$$K(\theta) = \inf \left\{ \frac{\mu(B(x, R))}{\mu(B(x, R/\theta))} \ : \ x \in \operatorname{supp}(\mu), R > 0 \right\}.$$

It then follows that a measure is uniformly perfect if and only if it has positive lower regularity dimension, with the lower regularity dimension again quantifying how uniformly perfect a measure is. Many of the previous references regarding the upper regularity dimension also discuss the lower regularity dimension; [HT19] is of particular interest.

Chapter 2

Upper and lower regularity dimensions

2.1 Introduction

This chapter aims to study some of the basic properties of the regularity dimensions. We investigate their relationship with familiar concepts, such as the local dimensions, the L^q-spectrum, and weak tangents. Then we will explicitly calculate the dimensions of self-similar measures satisfying the strong separation property and self-affine measures supported on carpets and sponges satisfying the very strong separation property, which are some of the most standard examples of fractals. Finally the dimensions of measures on convergent sequences will be studied. These examples will exhibit several different types of behaviour and will demonstrate the sharpness of our general results.

2.2 Results regarding bounds and examples

In this section we start by stating our results for the chapter. Bounds for the upper and lower regularity dimensions will be given in Section 2.2.1 whilst Section 2.2.2 will study the dimensions of weak tangent measures. The regularity dimensions of self-similar and self-affine measures will be calculated in Sections 2.2.3 and 2.2.4, respectively, whilst measures defined on certain sequences will be studied in Section 2.2.5. Proofs of these results will then follow in Section 2.3.

2.2.1 General bounds

The *local dimensions* of a measure are the main tool for characterising the local scaling laws of a measure at small scales. The upper local dimension of μ at

$x \in \text{supp}(\mu)$ is defined by

$$\overline{\dim}_{\text{loc}}(x,\mu) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

The lower local dimension $\underline{\dim}_{\text{loc}}(x,\mu)$ is defined similarly by

$$\underline{\dim}_{\text{loc}}(x,\mu) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

When these dimensions coincide we simply refer to the local dimension of the measure at x and write $\dim_{\text{loc}}(x,\mu)$. These local dimensions clearly depend on the point x, but naturally give rise to dimensions depending only on μ. For example, the lower Hausdorff dimension of μ is defined by

$$\underline{\dim}_{\text{H}}\mu = \operatorname{ess\,inf}\left\{\underline{\dim}_{\text{loc}}(x,\mu)\ :\ x \in \text{supp}(\mu)\right\}$$

and the upper packing dimension is

$$\overline{\dim}_{\text{P}}\mu = \operatorname{ess\,sup}\left\{\overline{\dim}_{\text{loc}}(x,\mu)\ :\ x \in \text{supp}(\mu)\right\}.$$

Recall, the essential supremum (ess sup) is the supremum which holds for a set of full measure, similarly for the essential infimum. For instance the Hausdorff dimension of a measure is often written

$$\underline{\dim}_{\text{H}}\mu = \sup\left\{s \geqslant 0\colon \underline{\dim}_{\text{loc}}(x,\mu) \geqslant s \text{ for } \mu \text{ almost all } x\right\}.$$

A measure μ is called exact dimensional when $\underline{\dim}_{\text{H}}\mu = \overline{\dim}_{\text{P}}\mu$. It follows from the above definitions that in this situation, almost all points have the same local scaling behaviour. This provides a significant level of regularity which is often sufficient when studying the Hausdorff dimension. However this does not imply that all points satisfy the same power law, and so the regularity dimensions, which focus on extremal points, need further conditions to ensure equality.

There is a fairly straightforward relationship between the upper regularity dimension of a measure and the upper local dimensions. One can immediately obtain the following

Lemma 1 ([FH20]). *For any locally finite Borel measure* μ

$$\overline{\dim}_{\text{reg}}\mu \geqslant \sup\left\{\overline{\dim}_{\text{loc}}(x,\mu)\ :\ x \in \text{supp}(\mu)\right\}.$$

Proof. In particular, let $x \in \text{supp}(\mu)$ and let $s > \overline{\dim}_{\text{reg}}\mu$. Then, by the definition of the upper regularity dimension, for small enough $r < R$ we have

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant C\left(\frac{R}{r}\right)^s$$

for some constant $C > 0$. Fixing R and letting $r \to 0$ we obtain

$$\overline{\dim}_{\mathrm{loc}}(x,\mu) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \leqslant \limsup_{r\to 0} \frac{\log \mu(B(x,R))(r/R)^s/C}{\log r} = s$$

as required. $\square$

This shows that the upper regularity dimension is sensitive to large upper local dimension even at a single point. This lower bound, combined with the Assouad dimension, gives a concrete and sometimes sharp lower bound on the upper regularity dimension. For example, we will show that for any self-similar measure μ satisfying the strong separation condition we have

$$\overline{\dim}_{\mathrm{reg}}\mu = \sup\left\{\overline{\dim}_{\mathrm{loc}}(x,\mu) \ : \ x \in \mathrm{supp}(\mu)\right\}.$$

However, for self-affine measures μ, one may have

$$\overline{\dim}_{\mathrm{reg}}\mu > \sup\left\{\overline{\dim}_{\mathrm{loc}}(x,\mu) \ : \ x \in \mathrm{supp}(\mu)\right\}.$$

Analogously, the lower regularity dimension of a measure μ can be related to the lower local dimensions by the following.

Lemma 2. *For any locally finite Borel measure μ supported on a metric space X*

$$\underline{\dim}_{\mathrm{reg}}\mu \leqslant \inf\left\{\underline{\dim}_{\mathrm{loc}}(x,\mu) \colon x \in \mathit{supp}(\mu)\right\}.$$

Proof. Indeed, let $x \in \mathrm{supp}(\mu)$ and $t < \underline{\dim}_{\mathrm{reg}}\mu$. Then for any $r < R$ small enough we have

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \geqslant C\left(\frac{R}{r}\right)^t$$

for some constant C. Fixing R and again letting $r \to 0$ gives

$$\underline{\dim}_{\mathrm{loc}}(x,\mu) = \liminf \frac{\log \mu(B(x,r)}{\log r} \geqslant t$$

as desired. $\square$

An exact dimensional measure μ is one for which $\underline{\dim}_{\mathrm{H}}\mu = \overline{\dim}_{\mathrm{P}}\mu$, that is the local dimension exists and is μ-almost everywhere constant. Such measures are commonly studied and have proved particularly helpful in the self-similar setting. As these two lemmas are sharp, we see that knowing a measure is exact dimensional is not sufficient to determine the regularity dimensions of that measure, self-similar measures will be an example of this behaviour. However, one can further restrict to Ahlfors-David regular measures to find examples where these lemmas suffice to calculate the regularity dimensions.

A measure μ is Ahlfors-David s-regular if there exists constants $0 < c_0, C_0 < \infty$ such that

$$c_0 R^s \leqslant \mu(B(x,R)) \leqslant C_0 R^s$$

for all $x \in \operatorname{supp}(\mu)$ and $0 < R < |\operatorname{supp}(\mu)|$. These measures are even more regular than exact dimensional measures in that the ratio $\mu(B(x,R))/R^s$ is uniformly bounded away from 0 and $+\infty$ where $s > 0$ is the 'dimension'. An Ahlfors-David s-regular measure is clearly exact dimensional (with exact dimension equal to s) and even satisfies $\underline{\dim}_{\mathrm{reg}}\mu = \overline{\dim}_{\mathrm{reg}}\mu = s$. This provides the first concrete examples of measures where the local dimensions can recover the regularity dimensions. Note that equality of the regularity dimensions does not imply that the measure is Ahlfors-David regular, see [HT19].

We have already noted that the Assouad dimension of the support and the supremum of the upper local dimensions give elementary lower bounds for the upper regularity dimension. We can refine this observation by further relating the upper regularity dimension to the *(lower) L^q-spectrum*. Let μ be a compactly supported probability measure on $\mathbb{R}^d$. Given $q \in \mathbb{R}$ and $r > 0$ we let

$$M_r^q(\mu) = \sup \left\{ \sum_{i=1}^{\infty} \mu(B(x_i,r))^q \ : \ d(x_i,x_j) > 2r \text{ for } i \neq j \right\}$$

be the multifractal packing function. The name comes from the fact that we are finding r-packings of the support of μ, that is collections of balls of radii r with centres in the support which are pairwise disjoint. Note we are constructing packings of the support of μ which is a compact set so this sum will be finite. See [Ols95] for further information about this function. The (lower) L^q-spectrum of μ is then given by

$$\underline{\tau}(q) = \liminf_{r\to 0} \frac{\log M_r^q(\mu)}{\log r}, \qquad (q \in \mathbb{R}).$$

The L^q-spectrum gives a description of the global fluctuations of the measure and is a key tool in multifractal analysis, along with the multifractal spectrum. The *multifractal spectrum* of μ is the function

$$f(\alpha) = \dim_{\mathrm{H}} \{x \in \operatorname{supp}(\mu) \colon \ \dim_{\mathrm{loc}}(x,\mu) = \alpha\}.$$

For example, the Legendre transform of the L^q-spectrum is an upper bound for the multifractal spectrum of μ and, for measures satisfying the multifractal formalism, the Legendre transform of the L^q-spectrum is precisely the multifractal spectrum. This is why q is taken to be any real in the definitions. Figure 4 illustrates this interatction. Further information on the multifractal formalism can be found in

[Ols95] and the references therein. As such, one can bound the supremum of the upper local dimensions by the 'top of the spectrum', defined by

$$T(\mu) = \sup\{s \geqslant 0 \colon \underline{\tau}(q) < sq \ , \forall q < 0\}$$

which is the gradient of the asymptote to $\underline{\tau}(q)$ as $q \to -\infty$. We prove that the upper regularity dimension is bounded below by the top of the spectrum.

Theorem 3. *Given a compactly supported Borel probability measure μ on $\mathbb{R}^d$, we have*

$$\begin{array}{ccccc} & \sup_{x\in \operatorname{supp}(\mu)} \overline{\dim}_{\mathrm{loc}}(x,\mu) & \leqslant & T(\mu) & \\ \rotatebox{-30}{$\geqslant$} & & & & \rotatebox{30}{$\leqslant$} \\ \overline{\dim}_{\mathrm{P}}\mu & & & & \overline{\dim}_{\mathrm{reg}}\mu. \\ \rotatebox{30}{$\leqslant$} & & & & \rotatebox{-30}{$\geqslant$} \\ & \dim_{\mathrm{P}} \operatorname{supp}(\mu) & \leqslant & \dim_{\mathrm{A}} \operatorname{supp}(\mu) & \end{array}$$

Most of the inequalities in the above theorem are known but are included to show the full picture. The new relation is between $T(\mu)$ and $\overline{\dim}_{\mathrm{reg}}\mu$ and was first shown in [FH20]; we prove it in Section 2.3.1. We take some inspiration from [FJ16] where the L^q-spectrum was related to the infimum of the lower local dimensions.

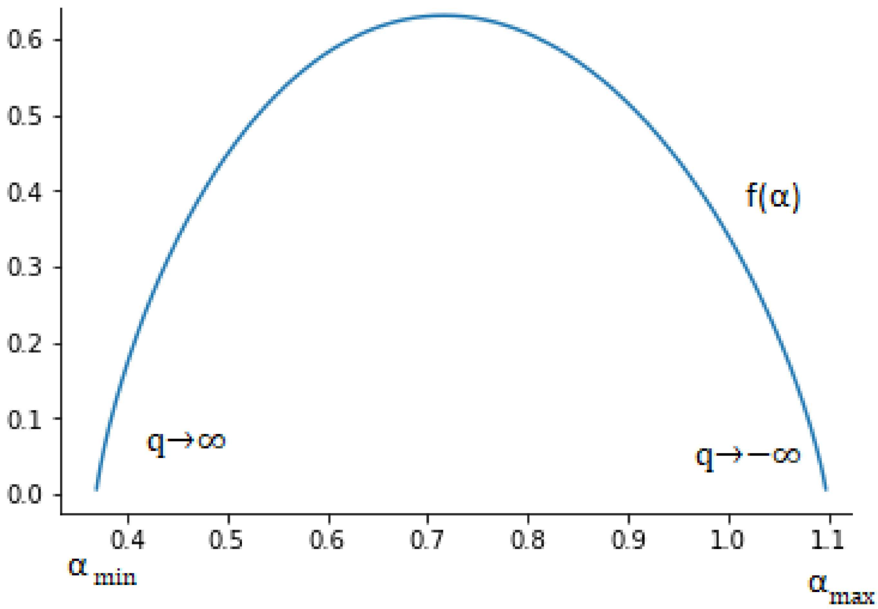


Figure 4: Interactions between several multifractal notions for a typical self-similar measure. Here the multifractal formalism holds, so $f(\alpha)$ coincides with the Legendre transform of the L^q-spectrum as indicated. $\alpha_{\min}$ and $\alpha_{\max}$ in this case coincide with the upper and lower regularity dimensions.

Expanding on the work of [FJ16], combined with the above ideas, gives bounds relating the lower local dimension, the 'lower end of the spectrum' and the lower regularity dimension. Formally, let the lower end of the spectrum be

$$t(\mu) = \inf\{s \geqslant 0 \colon \underline{\tau}(q) < sq \ , \forall q > 0\}\,.$$

Then the lower regularity dimension is bounded above by the lower end of the spectrum which is in turn always bounded above by the smallest local dimension.

Theorem 5. *Given a compactly supported Borel probability measure μ on $\mathbb{R}^d$, we have*

$$\underline{\dim}_{\mathrm{reg}}\mu \leqslant t(\mu) \leqslant \inf_{x \in \mathrm{supp}(\mu)} \underline{\dim}_{\mathrm{loc}}(x,\mu).$$

The inequality between $t(\mu)$ and $\inf_{x\in \mathrm{supp}(\mu)} \underline{\dim}_{\mathrm{loc}}(x,\mu)$ can be found in [Ols95, Lemma 4.4] whilst the proof of the second part of this relation will be in Section 2.3.1 and is new. As with the upper regularity dimension, these notions can be linked to the lower and Hausdorff dimensions using known results.

Many of the lower dimension results in this chater were first stated in [HT19], including the relation between the lower regularity dimension and the infimum of the lower local dimensions. However the proofs used in this thesis will follow the ideas found in [FH20] to ease comprehension.

2.2.2 Weak tangent measures

One of the most powerful tools for providing lower bounds for the Assouad dimension of a set is the well-known result of Mackay and Tyson concerning weak tangents [MT10, Proposition 6.1.5]. In particular, the Assouad dimension of any weak tangent to a set cannot exceed the Assouad dimension of the set itself. We state a slightly modified version of this result due to Fraser [Fra14, Proposition 7.7].

Proposition 6 (Very weak tangents)**.** *Let $X \subset \mathbb{R}^d$ be compact and let F be a compact subset of X. Let (T_k) be a sequence of bi-Lipschitz maps defined on $\mathbb{R}^d$ with Lipschitz constants $a_k, b_k \geqslant 1$ such that*

$$a_k|x-y| \leqslant |T_k(x) - T_k(y)| \leqslant b_k|x-y| \qquad (x, y \in \mathbb{R}^d),$$

where

$$\sup_k b_k/a_k = C_0 < \infty$$

and suppose that $T_k(F) \cap X \to \hat{F}$ in the Hausdorff metric. Then the set $\hat{F}$ is called a very weak tangent *to F and, moreover, $\dim_{\mathrm{A}} F \geqslant \dim_{\mathrm{A}} \hat{F}$.*

The Hausdorff distance between two non-empty compact subsets X, Y of $\mathbb{R}^d$ is defined by

$$d_{\mathcal{H}}(X,Y) = \max\left\{\sup_{x\in X}\inf_{y\in Y} d(x,y), \sup_{y\in Y}\inf_{x\in X} d(x,y)\right\}.$$

These weak tangents are often called microsets, as first defined by Furstenberg in the 60s [Fur70, Fur08], and have been fundamental in the study of the Assouad dimension. This includes aiding direct calculations of the dimension of sets as in [FH15], the direct relationship between the dimensions of microsets and the Assouad dimension as in [KOR16, FHKY19] and even linking fractal geometry with number theory and the Erdös-Turán conjecture in [FY16]. A result similar to Proposition 6 holds for the lower dimension and the smallest tangent sets with some added conditions; these additions were first introduced in [Fra14] but are not needed in the measure setting.

Given that the upper regularity dimension is the 'Assouad dimension of a measure', it is natural to consider a result analogous to Proposition 6 in the setting of measures, where we replace convergence in the Hausdorff metric with weak convergence of measures. A sequence of positive probability measures $\{\mu_n\}_{n\in\mathbb{N}}$ supported on subsets of a space X is said to converge weakly to another measure μ also supported on a subset of X if $\int f d\mu_n \to \int f d\mu$ as $n \to \infty$ for all continuous functions $f\colon X \to \mathbb{R}$. We then say that a measure $\hat{\mu}$ on $\mathbb{R}^d$ is a *weak tangent measure* to a locally finite measure μ on $\mathbb{R}^d$ if there exists a sequence of similarity maps T_k on $\mathbb{R}^d$ and a sequence of positive re-normalising numbers p_k such that

$$p_k \mu \circ T_k^{-1} \rightharpoonup \hat{\mu}$$

where $\rightharpoonup$ denotes weak convergence. Recall that a map T on $\mathbb{R}^d$ is a *similarity* if there is a constant $c \in (0,1)$ such that for all $x, y \in \mathbb{R}^d$ we have $d(T(x), T(y)) = c\, d(x,y)$. We refer to c as the *similarity ratio*.

Tangent measure usually refers to a weak limit of magnifications of a given measure at a specific point, in contrast to weak tangent measures where the point of magnification can move around. In the seminal work of Preiss [Pre87] tangent measures were allowed to have unbounded support and when one restricts the magnifications to a fixed compact set, the (compactly supported) limit measures are often called *micromeasures*, following the important work of Furstenberg [Fur08] and Hochman [Hoc10].

Theorem 7 ([FH20]). *Suppose $\hat{\mu}$ is a weak tangent measure to a locally finite Borel measure μ on $\mathbb{R}^d$. Then*

$$\overline{\dim}_{\mathrm{reg}}\hat{\mu} \leqslant \overline{\dim}_{\mathrm{reg}}\mu$$

and

$$\underline{\dim}_{\mathrm{reg}}\hat{\mu} \geqslant \underline{\dim}_{\mathrm{reg}}\mu.$$

We will prove this result in Section 2.3.2. Recall the definitions of doubling and unifromly perfect from equations (1.2) and (1.4) in the previous chapter. An immediate corollary is that weak tangent measures of doubling measures are doubling. This observation generalises the folklore result that 'tangent measures of doubling measures are doubling'. Similarly for uniformly perfect measures.

Corollary 8. *All weak tangent measures of doubling measures are doubling. All weak tangent measures of uniformly perfect measures are uniformly perfect.*

In our definition of weak tangent measures we follow the strategy of Preiss, allowing the tangent to have unbounded support. However the weak tangent (sets) for Assouad dimension follow the conventions of Furstenberg and Hochman, restricting to a compact set. It turns out that this difference in approach is necessary as the dimension of a set cannot increase under restriction but the dimension of a measure can. The general problem of determining when a doubling measure restricts to a doubling measure on a compact set of positive mass is a subtle problem, see [Oja14]. We now provide a simple example of a doubling measure on the plane which, upon restriction to the unit ball, becomes non-doubling; this examples was introduced in [FH20] to this end. In particular this shows that if, in the definition of weak tangent measures, one restricts the measures $p_k\mu \circ T_k^{-1}$ to the unit ball, the analogue of Theorem 7 generally fails. This is because the restricted measure would become a weak tangent measure to the original measure.

We begin with the square $X_0 = [-3/2, 3/2]^2$ and cut out a sequence of open 'almost' semicircles as follows. Let $\{r_i\}_{i\in\mathbb{N}}$ be a sequence of radii which decay exponentially to 0 and $\{x_i\}_{i\in\mathbb{N}}$ be a sequence of centres moving clockwise on S^1 which converge polynomially to a limit on the opposite side of S^1 from x_1 (without making any full rotations). Consider the balls $B_i = B(x_i, r_i)$ and remove from X_0 the points in the interior of $B_i \cap B(0,1)$ which are at distance strictly greater than r_i^2 from S^1, see the portion shown in grey on the left in Figure 9. After all of these portions have been removed, label the remaining set as X. The complement of X is shown in grey on the right of Figure 9.

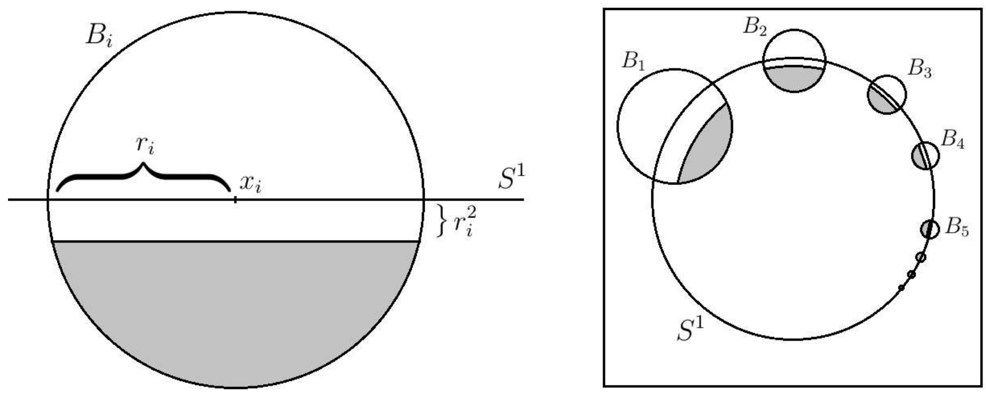


Figure 9: Construction of a doubling measure which is not doubling after restricting to the unit ball.

We observe that 2-dimensional Lebesgue measure μ on X is doubling, but that the restriction of μ to $B(0,1) \cap X$ is not doubling. It is easy to see that there exists a uniform constant $c > 0$ such that for $x \in X$ and $r \in (0,1)$ we have $cr^2 \leqslant \mu(B(x,r)) \leqslant \pi r^2$. It follows that μ is doubling (with upper regularity dimension 2) on X. We now consider the restricted measure $\nu = \mu|_{B(0,1)\cap X}$ and compare the masses of $B(x_i, r_i)$ and $B(x_i, 2r_i)$. For large enough i we have $\nu(B(x_i, 2r_i)) \geqslant (\pi(2r_i)^2 - \pi r_i^2)/3 = \pi r_i^2$ and $\nu(B(x_i, r_i)) \leqslant 4r_i^3$. Therefore

$$\frac{\nu(B(x_i, 2r_i))}{\nu(B(x_i, r_i))} \geqslant \frac{\pi r_i^2}{4r_i^3} = \frac{\pi}{4r_i} \to \infty$$

and so ν is not doubling.

2.2.3 Self-similar measures

In this section we compute the regularity dimensions of self-similar measures satisfying the strong separation condition. We emphasise that this separation assumption is natural because self-similar measures not satisfying the strong separation condition are typically not doubling and so have upper regularity dimension equal to $+\infty$. An example of this behaviour is provided below once the strong separation condition is formally defined.

Let $\mathcal{I}$ be a finite index set and $\{S_i\}_{i\in\mathcal{I}}$ be a finite collection of contraction maps on a compact subset of $\mathbb{R}^d$. Such a collection is known as an *iterated function system* (IFS). Also let $\{p_i\}_{i\in\mathcal{I}}$ be a collection of probabilities associated with the maps $\{S_i\}_{i\in\mathcal{I}}$, i.e. we assume that for each $i \in \mathcal{I}$ we have $p_i > 0$ and $\sum_{i\in\mathcal{I}} p_i = 1$. There is a unique non-empty compact set F satisfying

$$F = \bigcup_{i\in\mathcal{I}} S_i(F)$$

and a unique Borel probability measure μ satisfying

$$\mu = \sum_{i \in \mathcal{I}} p_i \mu \circ S_i^{-1}$$

which is fully supported on F, see [Fal03, Chapter 9] and the references therein, notably [Hut81]. When all of the contractions S_i are similarities, with similarity ratio $c_i \in (0,1)$, then F is called a *self-similar set* and μ is called a *self-similar measure*. These sets are some of the most well studied examples of fractals yet there are many subtleties that we still do not understand. A recent example is in [Bak19], where a self-similar set was constructed such that it does not contain exact overlaps, but does have super-exponentially close cylinders. We refer the reader to [Fal03] for a more in depth discussion of IFSs, self-similar sets and measures.

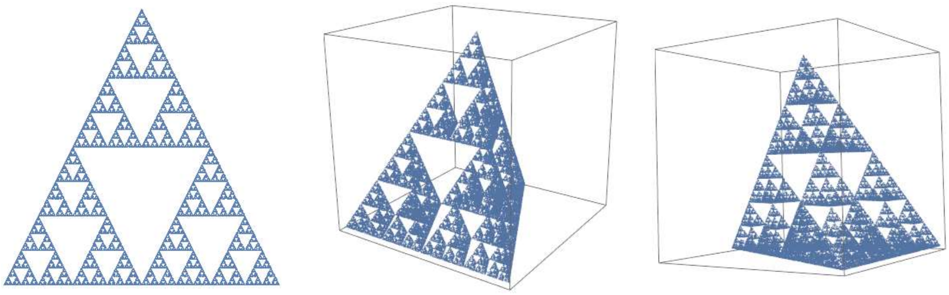


Figure 10: A self-similar Sierpiński triangle and two perspectives of a self-similar Sierpiński tetrahedron.

We say that the IFS (and associated set F and measure μ) satisfy the *strong separation condition (SSC)* if $S_i(F) \cap S_j(F) = \emptyset$ for all distinct $i, j \in \mathcal{I}$. This is a natural assumption in the context of the upper regularity dimension. For example, if the defining IFS consists of the maps $x \mapsto x/2$ and $x \mapsto x/2 + 1/2$, then the SSC is not satisfied and one easily verifies that μ is doubling if and only if both probabilities are equal to $1/2$ and in this case μ is Ahlfors-David 1-regular (it is Lebesgue measure on the unit interval). On the other hand, when the strong separation condition does not hold, measures can still have positive, even maximal lower regularity dimension; we do not pursue this here and direct the reader towards [HT19] for further elaboration.

It is a classical result due to [Hut81] that the Hausdorff dimension of a self-similar set satisfying the strong separation condition and with contractions $\{c_i\}_{i\in\lambda}$ is the unique s satisfying $\sum_{i\in\Lambda} c_i^s = 1$. In fact, these sets are incredibly regular so the box, Assouad and lower dimensions are all equal to the Hausdorff dimension. As we weaken the separation condition these numbers start to differ in a number of cases. Studying such cases has seen huge advances recently, as in [Hoc14].

The Hausdorff dimension of self-similar measures on self-similar sets satisfying the SSC is known to be $\underline{\dim}_{\mathrm{H}}\mu = \sum p_i \log p_i / \sum p_i \log c_i$ and these measures are exact dimensional. In this setting the multifractal formalism is satisfied and formulae for the ends of the spectrum can be found in [CM92, Ols95].

The regularity dimensions do not behave quite as nicely as one might hope in this setting in the sense that they do not equal the Hausdorff dimension of the measure. However, they do interact pleasantly with the multifractal spectrum of the measure.

Theorem 11. *Let μ be a self-similar measure as defined above and assume μ satisfies the SSC. Then*

$$\overline{\dim}_{\mathrm{reg}}\mu = \sup_x \overline{\dim}_{\mathrm{loc}}(x,\mu) = T(\mu) = \max_{i\in\mathcal{I}} \frac{\log p_i}{\log c_i}$$

and

$$\underline{\dim}_{\mathrm{reg}}\mu = \inf_x \overline{\dim}_{\mathrm{loc}}(x,\mu) = t(\mu) = \min_{i\in\mathcal{I}} \frac{\log p_i}{\log c_i}.$$

Calculations of the local dimensions and the ends of the spectrum in this setting are folklore and first appear in [CM92]. The upper regularity dimension was found in [FH20] and the lower regularity dimension in [HT19]

The Assouad dimension of the self-similar set which supports μ is generally strictly smaller than the upper regularity dimension of μ in this setting. In fact the only case where the geometric dimension and corresponding measure dimension coincide is when $p_i = c_i^s$, where s is the Hausdorff dimension of the set. In this case μ is Ahlfors-David s-regular and all of the notions of dimension for F and μ coincide and equal s.

The figure below is the first stage in the construction of a self-similar set in 2 dimensions with 3 maps f_1, f_2 and f_3 of respective contraction ratios 1/3, 1/3 and 1/2 and associated probabilities 1/6, 1/2 and 1/3. The attractor F of this IFS satisfies the strong separation condition so the main dimensions all coincide and equal

$$\dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = \dim_{\mathrm{A}} F = \dim_{\mathrm{L}} F \approx 1.17.$$

The probability vector and the IFS define a self-similar measure μ. The previously discussed dimensions of measures are distinct for this measure with

$$\underline{\dim}_{\mathrm{H}}\mu = \frac{4\log 2 + 3\log 3}{2\log 2 + 4\log 3} \approx 1.05,$$

$$\overline{\dim}_{\mathrm{reg}}\mu = \frac{\log 6}{\log 3} \approx 1.63$$

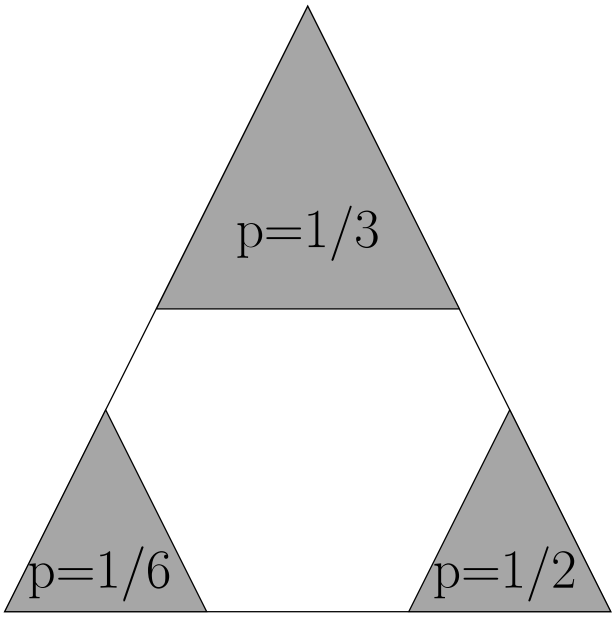


Figure 12: First step in the construction of a self-similar set satisfying the SSC and an associated measure.

and

$$\underline{\dim}_{\mathrm{reg}}\mu = \frac{\log 2}{\log 3} \approx 0.63.$$

In this section we restricted attention to self-similar sets satisfying the strong separation condition. Whilst certain overlap conditions have been covered in [HHT18], and others are already known, it is always of interest to extend to the most general setting possible.

Question 13. *What are the regularity dimensions of self-similar measures on sets not satisfying the strong separation condition?*

2.2.4 Self-affine measures

In this section we consider an important class of self-affine measures. Self-affine measures are defined in a similar way to the self-similar measures considered in the previous section, the only difference being that the defining contractions are assumed to be affinities rather than similarities. In general, such measures are much more difficult to handle due to the fact that different rates of distortion can occur in different directions. The specific class of self-affine measures we consider are those supported on Bedford-McMullen carpets, see [Bed84, McM84], and on the higher dimensional analogues, Bedford-McMullen sponges, see [KP96, Ols98].

The study of self-affine sets can be broken into two parts: the study of general self-affine sets and specific examples, such as Bedford-McMullen carpets. The generic case dates back to Falconer [Fal88] where the affinity dimension was first defined. This case has seen a great deal of study recently as in [BHR19, HR19]. On the other hand, studying specific examples has seen a number of different results, including [Bed84, McM84, LG92, Bar07, Mac10, Fra14]. In particular there is interest in understanding how the special cases, such as carpets, differ from the generic results, see [JM19, MS19].

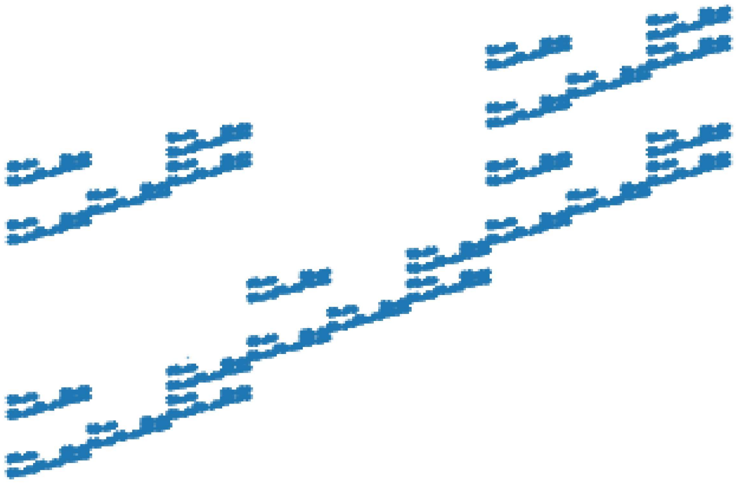


Figure 14: An example of a Bedford-McMullen carpet with a 3 by 4 grid and 5 maps.

Sponges (and measures defined on them) are less commonly studied, but came to prominence recently when they were used by Das and Simmons to provide a counterexample to an important and long-standing conjecture in dynamical systems [DS17]. In particular, there exists a surprising example of a sponge in $\mathbb{R}^3$ whose Hausdorff dimension cannot be approximated by the Hausdorff dimension of measures invariant under the natural associated dynamical system. This contrasts with the well known result of [Käe04] which states that an ergodic invariant measure can be found for typical self-affine sets so that the Hausdorff dimensions coincide.

Let $d \geqslant 2$ be an integer and fix integers $1 < n_1 < n_2 \cdots < n_d$. Choose a subset $\mathcal{I}$ of $\prod_{l=1}^{d} \{0, \ldots, n_l - 1\}$ and for $\mathbf{i} = (i_1, \ldots, i_d) \in \mathcal{I}$ let $S_{\mathbf{i}} \colon [0,1]^d \to [0,1]^d$ be defined by

$$S_{\mathbf{i}}(x_1, x_2 \ldots, x_d) = \left(\frac{x_1 + i_1}{n_1}, \frac{x_2 + i_2}{n_2}, \ldots, \frac{x_d + i_d}{n_d} \right).$$

Finally, consider the IFS $\{S_i\}_{i \in \mathcal{I}}$ acting on $[0,1]^d$ and let $\{p_i\}_{\mathbf{i} \in \mathcal{I}}$ be an associated probability vector as before. Let F be the associated attractor of this IFS, which is a self-affine set since each of the defining contractions is an affinity, and let μ be the associated self-affine measure, that is the unique Borel probability measure μ satisfying

$$\mu = \sum_{\mathbf{i} \in \mathcal{I}} p_{\mathbf{i}} \mu \circ S_{\mathbf{i}}^{-1}.$$

We can now state the separation condition we require, which we note is strictly stronger than the SSC.

Definition 15 (VSSC, [Ols98])**.** *A self-affine sponge F (associated to an index set $\mathcal{I}$) satisfies the* very strong separation condition (VSSC) *if the following condition holds. If $l \in \{1, \ldots, d\}$ and $(i_1, \ldots, i_d), (j_1, \ldots, j_d) \in \mathcal{I}$ satisfy $i_1 = j_1, \ldots, i_{l-1} = j_{l-1}$ and $i_l \neq j_l$, then $|i_l - j_l| > 1$.*

We assume the self-affine sets studied here satisfy the *very strong separation condition*, which was used by Olsen in [Ols98]. Again, this is the natural condition to assume in the context of regularity dimensions because without this assumption the self-affine measures tend not to be doubling, see [LWW16, FH15].

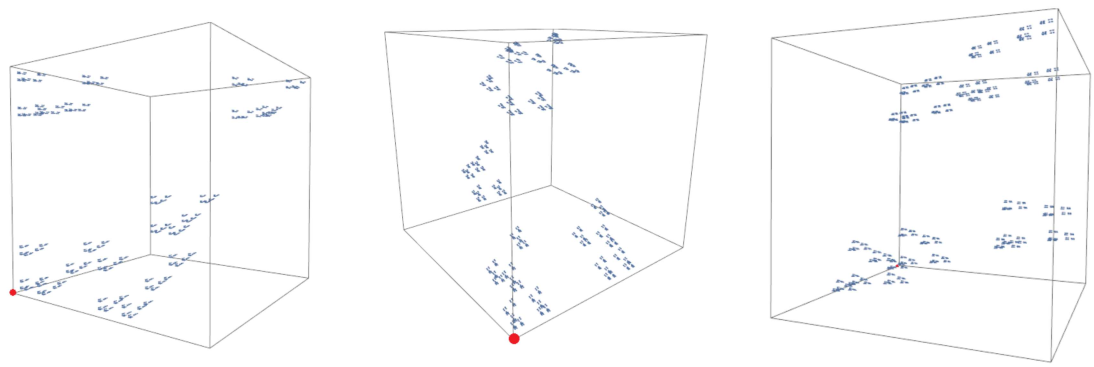


Figure 16: Three perspectives of the self-affine sponge defined by the data: $d = 3$, $n_1 = 3$, $n_2 = 4$, $n_3 = 5$ and $\mathcal{I} = \{(0,0,0), (0,2,0), (2,1,1), (2,3,4), (0,0,4)\}$. The origin is marked by a red dot to indicate orientation.

Before we state our result, we need to introduce some more notation. For $l = 1, \ldots, d$ and $\mathbf{i} = (i_1, \ldots, i_d) \in \mathcal{I}$ let

$$p_l(\mathbf{i}) = p(i_l | i_1, \ldots, i_{l-1}) = \frac{\displaystyle\sum_{\substack{\mathbf{j}=(j_1,\ldots,j_d) \in \mathcal{I} \\ j_1 = i_1, \ldots, j_{l-1} = i_{l-1}, j_l = i_l}} p_{\mathbf{j}}}{\displaystyle\sum_{\substack{\mathbf{j}=(j_1,\ldots,j_d) \in \mathcal{I} \\ j_1 = i_1, \ldots, j_{l-1} = i_{l-1}}} p_{\mathbf{j}}}$$

if $(i_1, \ldots, i_l, i_{l+1}, \ldots, i_d) \in \mathcal{I}$ for some $i_{l+1}, \ldots, i_d$ and 0 otherwise. These numbers have a clear interpretation: $p_l(\mathbf{i})$ is the conditional probability that the lth digit of an element of $\mathcal{I}$ coincides with the lth digit of $\mathbf{i}$, given that the first $l - 1$

coordinates did. Note that when $l = 1$ we are conditioning on the entire space and so the denominator of the above conditional probability is taken to be 1.

Theorem 17. *Let μ be a self-affine measure on a Bedford-McMullen sponge satisfying the VSSC. Then*

$$\overline{\dim}_{\mathrm{reg}}\mu = \sum_{l=1}^{d} \max_{\mathbf{i}\in\mathcal{I}} \frac{-\log p_l(\mathbf{i})}{\log n_l}$$

and

$$\underline{\dim}_{\mathrm{reg}}\mu = \sum_{l=1}^{d} \min_{\mathbf{i}\in\mathcal{I}} \frac{-\log p_l(\mathbf{i})}{\log n_l}.$$

The upper regularity dimension of self-affine measures on Bedford-McMullen sponges was first calculated in [FH20] and the lower dimension analogue was noted in [HT19].

For comparison, we will briefly consider the other notions of dimensions of Bedford-McMullen carpets, restricting to the two dimensional case for ease of notation. Let N be the total number of maps in the IFS and for each column $i \in \{0, \ldots, n_1 - 1\}$ we define N_i to be the number of maps in that specific column. Finally let N^0 to be the total number of non-empty columns. The Hausdorff and box dimensions of Bedford-McMullen carpets F are

$$\dim_{\mathrm{B}} F = \frac{\log N^0}{\log n_1} + \frac{\log N/N^0}{\log n_2}$$

and

$$\dim_{\mathrm{H}} F = \frac{\log \sum_{i=0}^{n_1-1} N_i^{\log n_1/\log n_2}}{\log n_1}.$$

The Assouad dimension of these sets was first calculated by Mackay in [Mac10], whilst the lower dimension is due to Fraser in [Fra14], and they are

$$\dim_{\mathrm{A}} F = \frac{\log N^0}{\log n_1} + \max_{i=0,\ldots,n_1-1} \frac{\log N_i}{\log n_2}$$

and

$$\dim_{\mathrm{L}} F = \frac{\log N^0}{\log n_1} + \min_{i=0,\ldots,n_1-1} \frac{\log \max\{N_i, 1\}}{\log n_2}.$$

Note the lower dimension has a $\max\{N_i, 1\}$ in the second fraction, this is to prevent well definededness issues when a column is empty so $N_i \neq 0$.

Formulae for the local dimensions $\sup_x \overline{\dim}_{\mathrm{loc}}(x, \mu), \inf_x \underline{\dim}_{\mathrm{loc}}(x, \mu)$ and spectrum $T(\mu), t(\mu)$ for the self-affine measures μ we consider in this section can be found in [Ols98], where the notation $\overline{a}, \underline{a}$ and $\overline{A}, \underline{A}$ was used, respectively.

We will now discuss a family of examples designed to demonstrate that all of the notions of dimension we discuss here can be distinct for self-affine measures. In particular, the upper regularity dimension can be strictly greater than the Assouad dimension, supremum of the upper local dimensions and the 'top of the spectrum', $T(\mu)$; similarly for the lower regularity dimension analogues. This behaviour was *not* seen in the self-similar case.

Let $d = 2$, $n_1 = 3$, $n_2 = 4$, $\mathcal{I} = \{(0, 2), (2, 1), (2, 3)\}$ and $p_{(0,2)} = \varepsilon$, $p_{(2,1)} = 1 - 3\varepsilon/2$ and $p_{(2,3)} = \varepsilon/2$ where we allow ε to vary in the interval $(0, 1/2]$. We write F for the self-affine carpet and μ for the self-affine measure associated with this data. Observe that the VSSC is satisfied and so our results apply. Theorem 17 yields

$$\overline{\dim}_{\mathrm{reg}}\mu = \frac{-\log \varepsilon}{\log 3} + \frac{-\log \frac{\varepsilon/2}{1-\varepsilon}}{\log 4}$$

and

$$\underline{\dim}_{\mathrm{reg}}\mu = \frac{-\log(1-\varepsilon)}{\log 3}.$$

Mackay's and Fraser's results give

$$\dim_{\mathrm{A}} F = \frac{\log 2}{\log 3} + \frac{\log 2}{\log 4}$$

and

$$\dim_{\mathrm{L}} F = \frac{\log 2}{\log 3}.$$

Olsen gives us

$$\sup_{x\in F} \overline{\dim}_{\mathrm{loc}}(x, \mu) = \max\left\{\frac{-\log \varepsilon}{\log 3}, \frac{-\log(1-\varepsilon)}{\log 3} + \frac{-\log \frac{\varepsilon/2}{1-\varepsilon}}{\log 4}\right\},$$

(which has a phase transition at $\varepsilon \approx 0.066$),

$$\inf_{x\in F} \underline{\dim}_{\mathrm{loc}}(x, \mu) = \min\left\{\frac{-\log \varepsilon}{\log 3}, \frac{-\log(1-\varepsilon)}{\log 3} + \frac{-\log \frac{1-3\varepsilon/2}{1-\varepsilon}}{\log 4}\right\},$$

$$T(\mu) = \frac{-\log \varepsilon}{\log 3} + \frac{\log 2}{\log 4}$$

and

$$t(\mu) = \frac{-\log 1-\varepsilon}{\log 3} + \frac{\log \frac{2(1-\varepsilon)}{1-3\varepsilon/2}}{\log 4}.$$

For $\varepsilon \in (0, 1/2)$ these quantities are all distinct and for $\varepsilon = 1/2$ the measure is the 'coordinate uniform measure' from [FH15] which has upper regularity dimension precisely equal to the Assouad dimension and lower regularity dimension equal to the lower dimension. This means the larger notions then become all equal and the lower concepts are also all the same, however the lower dimension is strictly less than the Assouad dimension so there is still a gap between them all.

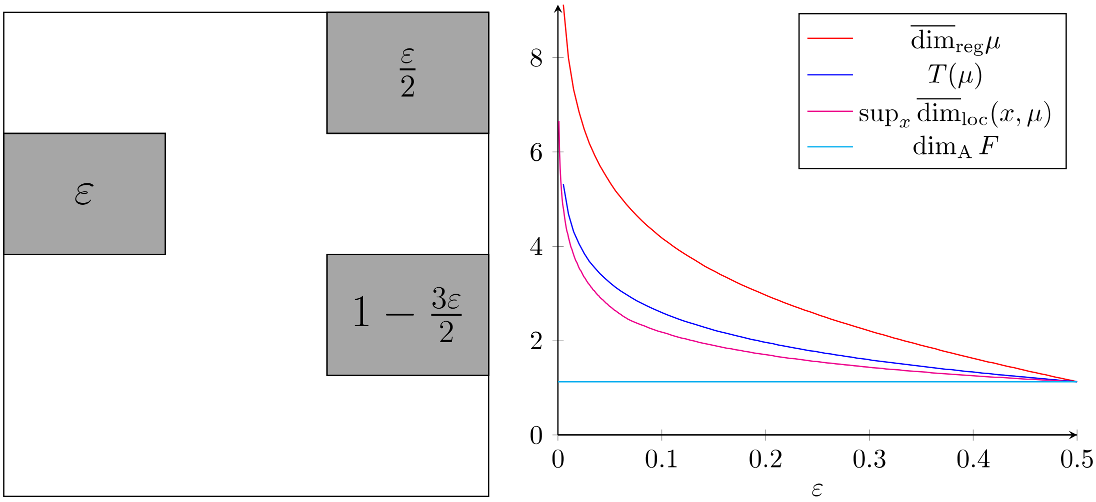


Figure 18: Left: the affine maps and associated probabilities. Right: a plot showing the four different 'upper dimensions' as ε varies.

As with self-similar measures, we have restricted to self-affine measures satisfying the very strong separation condition. This is natural and indeed necessary for full generality in the upper regularity dimension setting. It seems plausible to make general statements for the lower regularity dimension with weaker separation conditions and perhaps the upper regularity dimension can be computed in specific examples with weaker separation. We could also consider more general self-affine measures, such as self-affine measures on Lalley-Gatzouras carpets, whilst maintaining a strong separation condition; the set theoretic setting has a number of results in this direction which could be examined.

Question 19. *What can be said about the regularity dimensions of self-affine measures on sets which do not satisfy the VSSC? What are the regularity dimensions of self-affine measures on more general carpets, such as Lalley-Gatzouras carpets?*

2.2.5 Measures on sequences

When one first meets the Assouad and lower dimensions, the first interesting example is often that the set $\{1/n\}_{n\in\mathbb{N}}$ has Assouad dimension 1, which is strictly larger than the upper and lower box dimensions which are both 1/2, and lower dimension 0. Since this example is so prevalent, we decided to investigate the regularity dimensions of natural families of measures supported on such sets. For simplicity we restrict our examples to countable subsets of $[0, 1]$ with one accumulation point at 0 and measures equal to the sum of decaying point masses on the elements of the set. The interplay between the rate of convergence of the points in the set and the rate of decay of the point masses will turn out to be paramount to understanding the dimension of the measure, and to emphasise this we provide exact results for some simple cases where the rates of convergence are either polynomial or exponential. However, it could be interesting in the future to study sequences with other decay rates, for example stretched exponential decay a^{n^b} for $a, b \in (0,1)$, $n \in \mathbb{N}$.

More concretely, consider the set $\{x_n : n \in \mathbb{N}\}$ where $x_n \searrow 0$ and the sequence of weights $\{p(n) : n \in \mathbb{N}\}$ where $p(n) \searrow 0$, and $\sum_{n=1}^{\infty} p(n) < \infty$. The measure we are interested in is

$$\mu = \frac{1}{\sum_{n=1}^{\infty} p(n)} \sum_{n=1}^{\infty} p(n) \delta_{x_n}$$

where δ_{x_n} is a point mass at x_n.

Theorem 20 ([FH20]). *Let μ be as above.*

1. *Polynomial-polynomial: Let $\lambda > 0$ and $\omega > 1$ and suppose $x_n = n^{-\lambda}$ and $p(n) = n^{-\omega}$. Then*

$$\overline{\dim}_{\mathrm{reg}} \mu \;=\; \max\left\{1, \frac{\omega - 1}{\lambda}\right\} \;=\; \max\left\{\dim_{\mathrm{A}} \operatorname{supp}(\mu), \sup_x \overline{\dim}_{loc}(x, \mu)\right\}.$$

2. *Exponential-exponential: Let $\lambda, \omega \in (0,1)$ and suppose $x_n = \lambda^n$ and $p(n) = \omega^n$. Then*

$$\overline{\dim}_{\mathrm{reg}} \mu \;=\; \frac{\log \omega}{\log \lambda} \;=\; \sup_x \overline{\dim}_{\mathrm{loc}}(x, \mu) \;>\; 0 \;=\; \dim_{\mathrm{A}} \operatorname{supp}(\mu).$$

3. *Mixed rates: If*

   (i) $x_n = n^{-\lambda}$ $(\lambda > 0)$ and $p(n) = \omega^n$ $(0 < \omega < 1)$; or

   (ii) $x_n = \lambda^n$ $(0 < \lambda < 1)$ and $p(n) = n^{-\omega}$ $(\omega > 1)$,

   then μ is not doubling, and so $\overline{\dim}_{\mathrm{reg}} \mu = \infty$.

The above theorem can be summarised by the following table, for suitable values of ω and λ:

$p(n) \setminus x_n$	$n^{-\lambda}$	λ^n
$n^{-\omega}$	$\max\{1, \frac{\omega-1}{\lambda}\}$	∞
ω^n	∞	$\frac{\log \omega}{\log \lambda}$

The lower dimension of these sequences is always zero and, as the lower regularity dimension is a lower bound to the lower dimension, all of the previously discussed measures μ satisfy

$$\underline{\dim}_{\mathrm{reg}}\mu = 0.$$

Thus we have examples of doubling measures which are not uniformly perfect.

Here we see that having different rates for the set and measure always makes the measure non-doubling, investigating different rates of decay as mentioned previously might shed light on this phenomenon and could provide interesting results. It is worth noting that any new example should either have a consistent rate of decay for all its points or, at least, the gaps between points should decrease. Without these assumptions it is possible to build a sequence that behaves more like a sequence of self-similar sets than a genuine decreasing sequence of points, somewhat like the set constructed in the proof of Theorem 1.3 in [FHKY19]. Whilst this example is an interesting object in its own right, the techniques used here would not work in that setting so we avoid comparing the two.

Question 21. *How does the upper regularity dimension of measures defined on sequences behave for different decay rates, such as stretched exponential decay?*

2.3 Proofs

We prove Theorems 3 and 5 in Section 2.3.1 followed by Theorem 7 on weak tangents in Section 2.3.2. Section 2.3.3 will concern self-similar measures and will include a proof of Theorem 11. Self-affine measures and the proof of Theorem 17 will be dealt with in Section 2.3.4 along with some additional notation needed to study Bedford-McMullen sponges. Finally, Theorem 20 will be proved in Section 2.3.5. Any notation introduced in a subsection should only be used in that proof, but any notation used in Section 2.2 is assumed throughout the chapter.

2.3.1 Proof of Theorem 3: general relationships

We start this section with a proof of Theorem 3. Let μ be a probability measure supported on a compact set $X \subseteq \mathbb{R}^d$. Let $0 < s < T(\mu)$, $\overline{\dim}_{\mathrm{reg}}\mu < t < \infty$ and $q < 0$. By definition there exists a constant $C \geqslant 1$ such that for all $x \in X$ and for all $0 < r < 1$

$$\frac{\mu(B(x,1))}{\mu(B(x,r))} \leqslant C\left(\frac{1}{r}\right)^t.$$

In particular, this guarantees

$$\mu(B(x,r))^q \leqslant \frac{1}{C^q}\mu(B(x,1))^q r^{qt}$$

and, moreover,

$$M_r^q(\mu) \leqslant cr^{-d}r^{qt}$$

where $c > 0$ is a constant independent of r and where the r^{-d} term comes from an upper bound on r-packings of $X \subseteq \mathbb{R}^d$. Therefore

$$sq > \underline{\tau}(q) \geqslant qt - d$$

and so $s < t - d/q$ for any $q < 0$. By letting $q \to -\infty$ this yields $s \leqslant t$, which is sufficient to prove that $T(\mu) \leqslant \overline{\dim}_{\text{reg}}\mu$.

All that remains is to prove that $T(\mu) \geqslant \overline{\dim}_{\text{loc}}(x,\mu)$ for all $x \in X$. This was shown in [Ols95] where a finer result was even found; we present a simple proof of this result for completeness. Let $x \in X$ and $u > T(\mu)$ which implies that $\underline{\tau}(q) \geqslant uq$ for some $q < 0$ which we fix. Therefore, given $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$M_r^q(\mu) \leqslant C_\varepsilon r^{uq-\varepsilon}$$

for all $r \in (0,1)$. Since $\{B(x,r)\}$ is an r-packing of X, it follows that

$$\mu(B(x,r))^q \leqslant M_r^q(\mu) \leqslant C_\varepsilon r^{uq-\varepsilon}$$

and therefore

$$\mu(B(x,r)) \geqslant C_\varepsilon^{1/q} r^{u-\varepsilon/q}$$

for all $r \in (0,1)$, which proves that $\overline{\dim}_{\text{loc}}(x,\mu) \leqslant u - \varepsilon/q$ and since $\varepsilon > 0$ was arbitrary, this completes the proof of Theorem 3.

This technique can also be used to show Theorem 5. For a measure μ, let $s > t(\mu)$, $0 < t < \underline{\dim}_{\text{reg}}\mu$ and $q > 0$. From the definition of the lower regularity dimension, there exists a constant $C \geqslant 1$ such that for all $x \in X$ and all $0 < r < 1$

$$\frac{\mu(B(x,1))}{\mu(B(x,r))} \geqslant C\left(\frac{1}{r}\right)^t.$$

Thus, as q is now positive,

$$\mu(B(x,r))^q \leqslant C^q \mu(B(x,1))^q r^{tq}$$

and hence, as before,

$$M_r^q(\mu) \leqslant cr^{-d}r^{qt}$$

where c is a constant which does not depend on r.

Using $\underline{\tau}(q) = \liminf_{r\to 0} \frac{\log M_r^q(\mu)}{\log r}$ and the definition of $t(\mu)$ gives

$$sq > \underline{\tau}(q) \geqslant qt - d$$

and so

$$s > t - \frac{d}{q}.$$

Letting $q \to \infty$ gives $s \geqslant t$ as desired.

This final part can also be found in [Ols95, Lemma 4.4], we again provide a short proof. To show that $t(\mu) \leqslant \underline{\dim}_{\mathrm{loc}}(x, \mu)$ for all $x \in X$, let $u < t(\mu)$ so $\underline{\tau}(q) \geqslant uq$ for some $q > 0$ which is now fixed. Let $x \in X$ and $\varepsilon > 0$ so there exists a constant $C_\varepsilon > 0$ for which

$$M_r^q(\mu) \leqslant C_\varepsilon r^{uq-\varepsilon}$$

for all $r \in (0, 1)$. Again, as $\{B(x, r)\}$ is an r-packing of X, we see that

$$\mu(B(x, r))^q \leqslant C_\varepsilon r^{uq-\varepsilon}$$

and then, for all $r \in (0, 1)$,

$$\mu(B(x, r)) \leqslant C_\varepsilon^{1/q} r^{u-\varepsilon/q}.$$

Hence $\underline{\dim}_{\mathrm{loc}}(x, \mu) \geqslant u - \varepsilon/q$ and as ε was arbitrary, this completes the proof.

2.3.2 Weak tangent measures

Let μ be a locally finite measure on $\mathbb{R}^d$, $\{T_k\}_{k\in\mathbb{N}}$ a sequence of similarities on $\mathbb{R}^d$ with associated contraction ratios $\{c_k\}_{k\in\mathbb{N}}$, $\{p_k\}_{k\in\mathbb{N}}$ a sequence of positive renormalising numbers, and $\hat{\mu}$ be a corresponding weak tangent measure of μ, that is a measure on $\mathbb{R}^d$ such that

$$p_k \mu \circ T_k^{-1} \rightharpoonup \hat{\mu}$$

where $\rightharpoonup$ means weak convergence of measures. The Portmanteau Theorem (see [Mat95, Theorem 1.24]) says that this is equivalent to

$$\lim_{k\to\infty} p_k \mu \circ T_k^{-1}(A) = \hat{\mu}(A)$$

for all $\hat{\mu}$-continuity sets A. Recall that $A \subseteq \mathbb{R}^d$ is a $\hat{\mu}$-continuity set when $\hat{\mu}(\partial A) = 0$ with ∂A being the boundary of A. It is a simple exercise to show that for a fixed $x \in \mathbb{R}^d$, all but at most countably many balls $B(x, r)$ are $\hat{\mu}$-continuity sets of $\mathbb{R}^d$. Indeed, assume for a fixed $x \in X$, that there are uncountably many balls of different radii and centre x which are not $\hat{\mu}$-continuity sets. As $\hat{\mu}$ is locally finite, we can write it as a countable sum of finite measures $\hat{\mu}_i$. Since the sets $\partial B(x, r)$ are disjoint for distinct r, at most countably many $\partial B(x, r)$ can have positive $\hat{\mu}_i$ measure. Thus at most countably many $\partial B(x, r)$ can have positive $\hat{\mu}$

measure as desired.

We start by proving the upper regularity half of Theorem 7. A technical lemma is provided which reduces our calculation of the upper regularity dimension of $\hat{\mu}$ to the study of balls which are $\hat{\mu}$-continuity sets, this was first shown in [FH20].

Lemma 22. *Let ν be a locally finite measure on $\mathbb{R}^d$. Suppose there exist constants C and s such that for all $x \in \operatorname{supp}(\nu)$ and $0 < r < R$ such that $B(x,R)$ and $B(x,r)$ are ν-continuity sets, we have*

$$\frac{\nu(B(x,R))}{\nu(B(x,r))} \leqslant C\left(\frac{R}{r}\right)^s.$$

Then

$$\overline{\dim}_{\mathrm{reg}}\nu \leqslant s.$$

Proof. Assume

$$\frac{\nu(B(x,R))}{\nu(B(x,r))} \leqslant C\left(\frac{R}{r}\right)^s$$

holds for all ν-continuity balls. Fix $x \in \operatorname{supp}(\nu)$ and let $0 < r < R$ be arbitrary. Since there are at most countably many problematic radii, there must exist constants $\theta_1(x,r), \theta_2(x,R) \in [1,2]$ such that $B(x, \theta_1(x,r)^{-1}r)$ and $B(x, \theta_2(x,R)R)$ are ν-continuity balls. Thus

$$\frac{\nu(B(x,R))}{\nu(B(x,r))} \leqslant \frac{\nu(B(x,\theta_2(x,R)R))}{\nu(B(x,\theta_1(x,r)^{-1}r))} \leqslant C\left(\frac{\theta_2(x,R)R}{\theta_1(x,r)^{-1}r}\right)^s \leqslant 4^s C\left(\frac{R}{r}\right)^s$$

and it follows that $\overline{\dim}_{\mathrm{reg}}\nu \leqslant s$. □

We now return to proving Theorem 7. Let $x \in \operatorname{supp}(\hat{\mu})$ and $\rho > 0$ be such that $B(x,\rho)$ is a $\hat{\mu}$-continuity set. Therefore

$$\lim_{k\to\infty} p_k\mu \circ T_k^{-1}(B(x,\rho)) = \hat{\mu}(B(x,\rho)).$$

Thus, for sufficiently large k,

$$\frac{1}{2}p_k\mu \circ T_k^{-1}(B(x,\rho)) \leqslant \hat{\mu}(B(x,\rho)) \leqslant 2p_k\mu \circ T_k^{-1}(B(x,\rho)). \tag{2.1}$$

Let $\varepsilon > 0$ and $0 < r < R$ be such that both $B(x,R)$ and $B(x,r)$ are $\hat{\mu}$-continuity sets and choose k large enough so that (2.1) holds for $\rho = r$ and $\rho = R$. In particular,

$$\frac{\hat{\mu}(B(x,R))}{\hat{\mu}(B(x,r))} \leqslant 4\frac{p_k\mu \circ T_k^{-1}(B(x,R))}{p_k\mu \circ T_k^{-1}(B(x,r))} = 4\frac{\mu(T_k^{-1}(B(x,R)))}{\mu(T_k^{-1}(B(x,r)))}.$$

Note that T_k is a similarity of contraction ratio $c_k > 0$ and so $T_k^{-1}(B(x,R)) =$

$B(T_k^{-1}(x), c_k^{-1}R)$. Thus

$$\frac{\hat{\mu}(B(x,R))}{\hat{\mu}(B(x,r))} \leqslant 4\frac{\mu(B(T_k^{-1}(x), c_k^{-1}R))}{\mu(B(T_k^{-1}(x), c_k^{-1}r))}.$$

Here we wish to apply the definition of the upper regularity dimension of μ, but we cannot do this directly since $T_k^{-1}(x)$ does not have to be in $\mathrm{supp}(\mu)$. However, we can assume k is large enough (depending on r) so that there exists x' in the support of $\mu \circ T_k^{-1}$ which is at distance at most $r/2$ from x. Therefore

$$B(T_k^{-1}(x'), c_k^{-1}r/2) \subset B(T_k^{-1}(x), c_k^{-1}r)$$

and

$$B(T_k^{-1}(x), c_k^{-1}R) \subset B(T_k^{-1}(x'), 2c_k^{-1}R)$$

and $T_k^{-1}(x')$ is in the support of μ. Thus

$$\begin{aligned}\frac{\hat{\mu}(B(x,R))}{\hat{\mu}(B(x,r))} &\leqslant 4\frac{\mu(B(T_k^{-1}(x), c_k^{-1}R))}{\mu(B(T_k^{-1}(x), c_k^{-1}r))} \leqslant 4\frac{\mu(B(T_k^{-1}(x'), 2c_k^{-1}R))}{\mu(B(T_k^{-1}(x'), c_k^{-1}r/2))} \\ &\leqslant 4C\left(\frac{4c_k^{-1}R}{c_k^{-1}r}\right)^{\overline{\dim}_{\mathrm{reg}}\mu+\varepsilon} = 4^{1+\overline{\dim}_{\mathrm{reg}}\mu+\varepsilon}C\left(\frac{R}{r}\right)^{\overline{\dim}_{\mathrm{reg}}\mu+\varepsilon}\end{aligned}$$

where $C = C(\varepsilon) > 0$ is a uniform constant independent of x, r and R which comes from the definition of the upper regularity dimension of μ. Letting $\varepsilon \to 0$ proves $\overline{\dim}_{\mathrm{reg}}\hat{\mu} \leqslant \overline{\dim}_{\mathrm{reg}}\mu$ as desired.

For the lower dimension, a similar technique also holds. The following lemma is the lower regularity analogue to Lemma 22.

Lemma 23. *Let ν be a locally finite measure on $\mathbb{R}^d$. Suppose there exist constants C and s such that for all $x \in \mathrm{supp}(\nu)$ and $0 < r < R$ such that $B(x,R)$ and $B(x,r)$ are ν-continuity sets, we have*

$$\frac{\nu(B(x,R))}{\nu(B(x,r))} \geqslant C\left(\frac{R}{r}\right)^t.$$

Then $\underline{\dim}_{\mathrm{reg}}\nu \geqslant t$.

Proof. Assume

$$\frac{\nu(B(x,R))}{\nu(B(x,r))} \geqslant C\left(\frac{R}{r}\right)^t$$

holds for all ν-continuity balls. Fix $x \in \mathrm{supp}(\nu)$ and let $0 < r < R$ be arbitrary. As before there must exist constants $\theta_1(x,r), \theta_2(x,R) \in [1,2]$ such that

$B(x, \theta_1(x,r)r)$ and $B(x, \theta_2(x,R)^{-1}R)$ are ν-continuity balls. Thus

$$\frac{\nu(B(x,R))}{\nu(B(x,r))} \geqslant \frac{\nu(B(x,\theta_2(x,R)^{-1}R))}{\nu(B(x,\theta_1(x,r)r))} \geqslant 4^{-t}C\left(\frac{R}{r}\right)^t,$$

completing the proof. $\square$

It again suffices to prove the second part of Theorem 7 for $\hat{\mu}$-continuity sets. Let $x \in \text{supp}(\hat{\mu})$, $\varepsilon > 0$ and $0 < r < R$ be such that both $B(x,R)$ and $B(x,r)$ are $\hat{\mu}$-continuity sets and choose k large enough so that (2.1) holds for $\rho = r$ and $\rho = R$. Thus,

$$\frac{\hat{\mu}(B(x,R))}{\hat{\mu}(B(x,r))} \geqslant 4^{-1}\frac{\mu(B(T_k^{-1}(x), c_k^{-1}R))}{\mu(B(T_k^{-1}(x), c_k^{-1}r))}.$$

As before, $T_k^{-1}(x)$ does not have to be in $\text{supp}(\mu)$ so the definition of the lower regularity dimension can not be applied directly. However, by choosing k large enough (depending on r) there exists x' in the support of $\mu \circ T_k^{-1}$ such that

$$B(T_k^{-1}(x), c_k^{-1}r) \subset B(T_k^{-1}(x'), 2c_k^{-1}r)$$

and

$$B(T_k^{-1}(x'), c_k^{-1}R/2) \subset B(T_k^{-1}(x), c_k^{-1}R).$$

Therefore

$$\begin{aligned}\frac{\hat{\mu}(B(x,R))}{\hat{\mu}(B(x,r))} &\geqslant 4^{-1}\frac{\mu(B(T_k^{-1}(x'), c_k^{-1}R/2))}{\mu(B(T_k^{-1}(x'), 2c_k^{-1}r))} \\ &\geqslant (4^{-1})^{1+\underline{\dim}_{\mathrm{reg}}\mu+\varepsilon}C\left(\frac{R}{r}\right)^{\underline{\dim}_{\mathrm{reg}}\mu+\varepsilon}\end{aligned}$$

where $C = C(\varepsilon) > 0$ is the constant in the definition of the lower regularity dimension of μ. As ε was arbitrary, this proves $\underline{\dim}_{\mathrm{reg}}\hat{\mu} \geqslant \underline{\dim}_{\mathrm{reg}}\mu$ as desired.

2.3.3 Proof of Theorem 11: self-similar measures

If F is a self-similar set defined with the IFS $\{S_i\}_{i\in\mathcal{I}}$, there is a natural correspondence between the geometric fractal F and the symbolic space $\mathcal{I}^{\mathbb{N}}$ (the set of all infinite words over $\mathcal{I}$) via the coding map $\pi\colon \mathcal{I}^{\mathbb{N}} \to \mathbb{R}^d$ defined by

$$\{\pi(i_0, i_1, \ldots)\} = \bigcap_{n=0}^{\infty} S_{i_0,\ldots,i_n}(F)$$

where $S_{i_0,\ldots,i_n} = S_{i_0} \circ \cdots \circ S_{i_n}$. This technique is useful as the symbolic space is often easier to work with than the Euclidean geometry of the attractor and since π provides a natural passage between the two settings we can work in the space that is most helpful whenever desired.

For $i_0, i_1, \ldots, i_{n-1} \in \mathcal{I}$ we define a cylinder $[i_0, i_1, \ldots, i_{n-1}] \subseteq \mathcal{I}^{\mathbb{N}}$, to be the set of all words in $\mathcal{I}^{\mathbb{N}}$ whose first n letters are $i_0, i_1, \ldots, i_{n-1}$. The collection of all level n cylinders corresponds to the n'th level pre-fractal of the attractor, an approximation of the attractor itself. Due to the construction, it can be seen that $F = \pi(\mathcal{I}^{\mathbb{N}})$ and the SSC also guarantees that π is a bijection so we may interchange the symbolic and geometric spaces, see [Fal03] for further details. Also note that $c_{i_0} \cdots c_{i_n}$ is the contraction ratio of the similarity $S_{i_0,\ldots,i_n}$, $\pi([i_0, i_1, \ldots, i_n]) = S_{i_0,\ldots,i_n}(F)$ and the μ measure of $S_{i_0,\ldots,i_n}(F)$ is $p_{i_0} \cdots p_{i_n}$. By rescaling if necessary, we may assume without loss of generality that $|F| = 1$.

We define δ to be the minimal distance between distinct sets $S_i(F)$ and $S_j(F)$, that is

$$\delta = \min_{i \neq j} \inf_{\substack{x \in S_i(F) \\ y \in S_j(F)}} d(x, y).$$

The SSC guarantees that $\delta > 0$. Thus the minimal distance between $S_{i_0,\ldots,i_{n-1},i}(F)$ and $S_{i_0,\ldots,i_{n-1},j}(F)$ is at least $c_{i_0} \cdots c_{i_{n-1}} \delta$ for any $i_0, i_1, \ldots, i_{n-1} \in \mathcal{I}$ and $i \neq j$.

For $x \in F$ with $\pi(i_0, i_1, \ldots) = x$ for a unique $(i_0, i_1, \ldots) \in \mathcal{I}^{\mathbb{N}}$ and small $r > 0$, we define the integer $n(x, r)$ to be the largest integer such that $r \leqslant c_{i_0} c_{i_1} \cdots c_{i_{n(x,r)}}$ and so

$$c_{i_0} c_{i_1} \cdots c_{i_{n(x,r)+1}} < r \leqslant c_{i_0} c_{i_1} \cdots c_{i_{n(x,r)}}.$$

We also let $m(x, r)$ be the smallest non-negative integer such that

$$\pi([i_0, \ldots, i_{m(x,r)}]) \subset B(x, r/2),$$

and so, in particular, $p_{i_0} \cdots p_{i_{m(x,r)}} \leqslant \mu(B(x, r))$. Note that for any $x \in F$, and small $r > 0$, $c_{i_0} \cdots c_{i_{m(x,r)}} \leqslant r$. By choosing $m(x, r)$ to be minimal, $\pi([i_0, \ldots, i_{m(x,r)-1}]) \not\subset B(x, r/2)$ and so $c_{i_0} \cdots c_{i_{m(x,r)-1}} \geqslant r/2$. Using the SSC, we have that for all $x = \pi(i_0, \ldots)$ and $r > 0$, we have $\mu(B(x, \frac{\delta r}{2})) \leqslant p_{i_0} \cdots p_{i_{n(x,r)}}$ where δ is the separation constant determined by the SSC. This is true since $B(x, \frac{\delta r}{2}) \cap F \subseteq \pi([i_0, \ldots, i_{n(x,r)}])$.

We are now in a position to prove both parts of Theorem 11. The setup so far is sufficient for both dimension results and the proofs are alike. We start by proving a lower bound to the upper regularity dimension. As mentioned previously, this follows directly from the calculations in [CM92] but the proof is short and so we include it for clarity.

Let $x = \pi(\omega_{\max})$ where $\omega_{\max} \in \mathcal{I}^{\mathbb{N}}$ is an infinite string of the symbol i which maximises $\log p_i/\log c_i$. It follows that $n(x,r) > \log r/\log c_i - 1$ and therefore

$$\mu(B(x,\delta r/2)) \leqslant p_i^{n(x,r)} \leqslant p_i^{-1} r^{\log p_i/\log c_i}$$

and it follows that $\overline{\dim}_{\text{loc}}(x,\mu) \geqslant \log p_i/\log c_i =: s$. Moreover, Theorem 3 yields $\overline{\dim}_{\text{reg}}\mu \geqslant \sup_{x\in F} \overline{\dim}_{\text{loc}}(x,\mu) \geqslant s$.

We will now demonstrate the reverse inequality. As F satisfies the SSC and μ is a self-similar measure, it is doubling (see [Ols95], for example). Thus recall there exists a constant $C(2/\delta) \geqslant 1$ depending only on $\delta/2 < 1$ such that $\frac{\mu(B(x,R))}{\mu(B(x,\delta R/2))} \leqslant C(2/\delta)$ for any $x \in F$ and for any $R > 0$. Let $x \in F$ and $0 < r < R$ and assume without meaningful loss of generality that $n(x,R) < m(x,r)$. If this were not true, then R/r is bounded above by a uniform constant – a situation we can safely ignore. Hence

$$\begin{aligned}
\frac{\mu(B(x,R))}{\mu(B(x,r))} &\leqslant C(2/\delta)\frac{\mu(B(x,\delta R/2))}{\mu(B(x,r))} \\
&\leqslant C(2/\delta)\frac{p_{i_0}\cdots p_{i_{n(x,R)}}}{p_{i_0}\cdots p_{i_{m(x,r)}}} \\
&= C(2/\delta)\frac{1}{p_{i_{n(x,R)+1}}}\cdots\frac{1}{p_{i_{m(x,r)}}} \\
&= C(2/\delta)\left(\frac{1}{c_{i_{n(x,R)+1}}}\right)^{\frac{\log p_{i_{n(x,R)+1}}}{\log c_{i_{n(x,R)+1}}}}\cdots\left(\frac{1}{c_{i_{m(x,r)}}}\right)^{\frac{\log p_{i_{m(x,r)}}}{\log c_{i_{m(x,r)}}}} \\
&\leqslant C(2/\delta)\left(\frac{1}{c_{i_{n(x,R)+1}}}\right)^{s}\cdots\left(\frac{1}{c_{i_{m(x,r)}}}\right)^{s} \\
&= C(2/\delta)\left(\frac{1}{c_{i_{n(x,R)+1}}c_{i_{m(x,r)}}}\right)^{s}\left(\frac{c_{i_0}c_{i_1}\cdots c_{i_{n(x,R)+1}}}{c_{i_0}c_{i_1}\cdots c_{i_{m(x,r)-1}}}\right)^{s} \\
&\leqslant C(2/\delta)\left(\frac{2}{c_{\min}^2}\right)^{s}\left(\frac{R}{r}\right)^{s}
\end{aligned}$$

where $c_{\min} = \min_{i\in\mathcal{I}} c_i$, is just a constant. The desired upper bound, and the first part of Theorem 11, follows.

To find an upper bound for the lower regularity dimension, choose $x = \pi(\omega_{\min})$ with $\omega_{\min} \in \mathcal{I}^{\mathbb{N}}$ the infinite string of the repeated symbol i which minimises $\log p_i/\log c_i$. Then $n(x,r) \leqslant \log r/\log c_i$ and thus

$$\mu(B(x,\delta r/2)) \geqslant r^{\log p_i/\log c_i}.$$

Hence $\underline{\dim}_{\text{loc}}(x,\mu) \leqslant \log p_i/\log c_i =: t$ and $\underline{\dim}_{\text{reg}}\mu \leqslant \inf_{x\in F}\underline{\dim}_{\text{loc}}(x,\mu) \leqslant t$ by Theorem 5 as desired. As with the lower bound for the upper regularity dimension, this can also be seen from [CM92].

The lower bound follows in the same way that the upper bound for the upper regularity dimension was found, we summarise this calculation. Note that the measure is still doubling and the previous inequality regarding the doubling constant will be used here, this is simply to make the choice of $n(x, r)$ more natural.

$$
\begin{aligned}
\frac{\mu(B(x,R))}{\mu(B(x,r))} &\geqslant C(2/\delta)^{-1}\frac{\mu(B(x,R))}{\mu(B(x,\delta r/2))} \\
&\geqslant C(2/\delta)^{-1}\frac{p_{i_0}\cdots p_{i_{m(x,R)}}}{p_{i_0}\cdots p_{i_{n(x,r)}}} \\
&= C(2/\delta)^{-1}\left(\frac{1}{c_{i_{m(x,R)+1}}}\right)^{\log p_{i_{m(x,R)+1}}/\log c_{i_{m(x,R)+1}}}\cdots\left(\frac{1}{c_{i_{n(x,r)}}}\right)^{\frac{\log p_{i_{n(x,r)}}}{\log c_{i_{n(x,r)}}}} \\
&\geqslant C(2/\delta)^{-1}\left(\frac{1}{c_{i_{m(x,R)+1}}}\right)^t\cdots\left(\frac{1}{c_{i_{n(x,r)}}}\right)^t \\
&= C(2/\delta)^{-1}\left(\frac{c_{i_0}c_{i_1}\cdots c_{i_{m(x,R)}}}{c_{i_0}c_{i_1}\cdots c_{i_{n(x,r)}}}\right)^t \\
&\geqslant C(2/\delta)^{-1}\left(\frac{c_{i_{m(x,R)}}c_{i_{n(x,r)}}}{2}\right)^t\left(\frac{R}{r}\right)^t \geqslant C(2/\delta)^{-1}\left(\frac{c_{\min}^2}{2}\right)^t\left(\frac{R}{r}\right)^t
\end{aligned}
$$

where $c_{\min}$ is as before, completing the proof.

2.3.4 Proof of Theorem 17: self-affine measures

Similar to the previous section, we use the natural correspondence between the self-affine set $F \subseteq \mathbb{R}^d$ and the symbolic space $\mathcal{I}^{\mathbb{N}}$ (the set of all infinite words over $\mathcal{I}$) via the coding map $\pi\colon \mathcal{I}^{\mathbb{N}} \to \mathbb{R}^d$ defined by

$$\{\pi(\mathbf{i}_1, \mathbf{i}_2, \ldots)\} = \bigcap_{n=1}^{\infty} S_{\mathbf{i}_1,\ldots,\mathbf{i}_n}(F).$$

where $S_{\mathbf{i}_1,\ldots,\mathbf{i}_n} = S_{\mathbf{i}_1} \circ \cdots \circ S_{\mathbf{i}_n}$. Recall that elements of $\mathcal{I}$ have d coordinates so we write $(\mathbf{i}_1, \ldots) = ((i_{1,1}, \ldots, i_{1,d}), \ldots) \in \mathcal{I}^{\mathbb{N}}$.

Since the cylinders scale by different amounts in different directions, they do not directly approximate a ball in measure. For this reason, we introduce *approximate cubes*. For $r \in (0, 1]$, choose the unique integers $k_1(r), \ldots, k_d(r)$, greater than or equal to 0, satisfying

$$\frac{1}{n_l^{k_l(r)+1}} < r \leqslant \frac{1}{n_l^{k_l(r)}}$$

for $l = 1, \ldots, d$. In particular,

$$\frac{-\log r}{\log n_l} - 1 < k_l(r) \leqslant \frac{-\log r}{\log n_l}. \tag{2.2}$$

Then the approximate cube $Q(\omega, r)$ of (approximate) side length r determined by $\omega = (\mathbf{i}_1, \mathbf{i}_2, \ldots) = ((i_{1,1}, \ldots, i_{1,d}), (i_{2,1}, \ldots, i_{2,d}), \ldots) \in \mathcal{I}^{\mathbb{N}}$ is defined by

$$\begin{aligned} Q(\omega, r) = \big\{\omega' = (\mathbf{j}_1, \mathbf{j}_2, \ldots) \in \mathcal{I}^{\mathbb{N}} : \forall\ l = 1, \ldots, d& \\ \text{and } \forall\ t = 1, \ldots, k_l(r) \text{ we have } j_{t,l} = i_{t,l}\big\}&. \end{aligned}$$

The geometric analogue is $\pi\left(Q(\omega, r)\right)$, which is contained in

$$\prod_{l=1}^{d} \left[\frac{i_{1,l}}{n_l} + \cdots + \frac{i_{k_l(r),l}}{n_l^{k_l(r)}}, \frac{i_{1,l}}{n_l} + \cdots + \frac{i_{k_l(r),l}}{n_l^{k_l(r)}} + \frac{1}{n_l^{k_l(r)}}\right];$$

a hypercuboid in $\mathbb{R}^d$ aligned with the coordinate axes with side lengths $n_l^{-k_l(r)}$, which are all comparable to r since $r \leqslant n_l^{-k_l(r)} < n_l r$. Thus the measure of a ball can be closely approximated by the measure of an appropriate approximate cube. This is made precise by the following useful proposition due to Olsen [Ols98, Proposition 6.2.1].

Proposition 24 ([Ols98])**.** *Let $\omega \in \mathcal{I}^{\mathbb{N}}$ and $k \in \mathbb{N}$.*

1. *If the VSSC is satisfied, then $B\left(\pi(\omega), 2^{-1}n_1^{-k}\right) \cap F \subseteq \pi\left(Q\left(\omega, n_1^{-k}\right)\right)$.*
2. *$\pi\left(Q\left(\omega, n_1^{-k}\right)\right) \subseteq B\left(\pi(\omega), (n_1 + \cdots + n_d)n_1^{-k}\right)$.*

This proposition means, since we assume the VSSC, that a ball of a particular radius contains, and is contained in, an approximate cube of a comparable radius. Therefore we may replace balls with approximate cubes in the definition of the regularity dimensions, which makes the calculations much easier. This method was used in [FH15, Proposition 3.1 and 3.5] to calculate the Assouad and lower dimensions of sponges.

Formally, for any $\omega \in \mathcal{I}^{\mathbb{N}}$ with $x = \pi(\omega) \in F$ and $0 < R \leqslant 1$, by Proposition 24 we see that

$$\pi(Q(\omega, \frac{R}{n_1(n_1 + \cdots + n_d)})) \subseteq B(x, R) \subseteq \pi(Q(\omega, 2n_1 R)).$$

Thus

$$\frac{\mu\left(\pi\left(Q\left(\omega, \frac{R}{n_1(n_1+\cdots+n_d)}\right)\right)\right)}{\mu(\pi(Q(\omega, 2n_1 r)))} \leqslant \frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant \frac{\mu(\pi(Q(\omega, 2n_1 R)))}{\mu\left(\pi\left(Q\left(\omega, \frac{r}{n_1(n_1+\cdots+n_d)}\right)\right)\right)}$$

and so to compute $\overline{\dim}_{\text{reg}}\mu$ and $\underline{\dim}_{\text{reg}}\mu$ it suffices to consider

$$\frac{\mu(\pi(Q(\omega,R)))}{\mu(\pi(Q(\omega,r)))}$$

directly.

Recalling the conditional probabilities $p(i_l|i_1,\dots,i_{l-1})$, defined in Section 2.2.4, which give the probability of having i_l as the 1^{th} coordinate given the previous ones, we can write down an explicit formula for the μ measure of an approximate cube for any $\omega \in \mathcal{I}^{\mathbb{N}}$ and $r \in (0,1)$:

$$\mu(\pi(Q(\omega,r))) = \prod_{l=1}^{d} \prod_{j=0}^{k_l(r)-1} p_l(\sigma^j\omega) \tag{2.3}$$

where $p_l(\omega) = p(i_{1,l}|i_{1,1},\dots,i_{1,l-1})$ and $\sigma : \mathcal{I}^{\mathbb{N}} \to \mathcal{I}^{\mathbb{N}}$ is the left shift. This formula follows immediately from the definition of μ as a self-affine measure on a Bedford-McMullen sponge and was first observed by Olsen [Ols98, Equation 6.2].

With the preliminaries out of the way we are now able to tackle the upper regularity dimension. For $l = 1,\dots,d$, let $p_l^{\min} = \min_{j\in\mathcal{I}} p_l(\mathbf{j})$ and let $i_l^{\min} \in \mathcal{I}$ be an element achieving this minimum. If such an element is not unique, it does not matter which we choose. Let $s = \sum_{l=1}^{d} \frac{-\log p_l^{\min}}{\log n_l}$ be the target dimension. We begin with an upper bound using (2.3). Let $x = \pi(\omega) \in F$ and $0 < r < R \leqslant 1$ and for convenience we assume without loss of generality that $r < R/n_d$, which ensures $k_l(R) < k_l(r)$ for all l. Recalling equation (2.2) and noting that $a^{\log b} = b^{\log a}$ for any $a, b > 0$, we have

$$
\begin{aligned}
\frac{\mu(\pi(Q(\omega,R)))}{\mu(\pi(Q(\omega,r)))} &= \frac{\prod_{l=1}^{d}\left(\prod_{j=0}^{k_l(R)-1} p_l(\sigma^j\omega)\right)}{\prod_{l=1}^{d}\left(\prod_{j=0}^{k_l(r)-1} p_l(\sigma^j\omega)\right)} = \frac{1}{\prod_{l=1}^{d}\left(\prod_{j=k_l(R)}^{k_l(r)-1} p_l(\sigma^j\omega)\right)} \\
&\leqslant \prod_{l=1}^{d} \frac{1}{\prod_{j=k_l(R)}^{k_l(r)-1} p_l^{\min}} \\
&= \prod_{l=1}^{d} \left(\frac{1}{p_l^{\min}}\right)^{k_l(r)-k_l(R)} \\
&\leqslant \prod_{l=1}^{d} \left(\frac{1}{p_l^{\min}}\right)^{-\log r/\log n_l + \log R/\log n_l + 1} \\
&\leqslant p^{-d}\left(\frac{R}{r}\right)^{s}
\end{aligned}
$$

where $p = \min_l p_l^{\min} > 0$ is a constant. It follows that $\overline{\dim}_{\mathrm{reg}}\mu \leqslant s$.

For the lower bound, we aim to find a sequence of points in F and scales $r < R$ for which the ratio of measures will behave like $(R/r)^s$. We have to be a little more careful with the relationship between R and r in this setting though. Again we assume that $r < R/n_d$, which ensures $k_l(R) < k_l(r)$ for all l. However, for technical reasons we also require $k_l(R) > k_{l+1}(r)$ for all $l = 1, \ldots, d-1$. For this it is sufficient to assume

$$(n_l R)^{\frac{\log n_{l+1}}{\log n_l}} < r$$

and fortunately we can choose r satisfying both of these conditions simultaneously. This is where we use the fact that the n_l are strictly increasing. Moreover, we can choose a sequence of pairs (r, R) such that $R \to 0$, $R/r \to \infty$ and

$$(n_l R)^{\frac{\log n_{l+1}}{\log n_l}} < r < R/n_d.$$

Let R, r be a pair from this sequence and observe that

$$k_d(R) < k_d(r) < k_{d-1}(R) < k_{d-1}(r) < \cdots < k_2(R) < k_2(r) < k_1(R) < k_1(r). \tag{2.4}$$

For fixed r, R as above, let $\omega \in \mathcal{I}^{\mathbb{N}}$ (and so $x = \pi(\omega)$) be chosen such that

$$
\begin{aligned}
\omega = (&i_1, \ldots, i_{k_d(R)}, i_d^{\min}, \ldots, i_d^{\min}, i_{k_d(r)+1}, \ldots, i_{k_2(R)}, \\
&i_2^{\min}, \ldots, i_2^{\min}, i_{k_2(r)+1}, \ldots, i_{k_1(R)}, i_1^{\min}, \ldots, i_1^{\min}, i_{k_1(r)+1}, \ldots),
\end{aligned}
$$

where the coordinates not specified as $i_l^{\min}$ (for some $l \in \{1, \ldots, d\}$) are arbitrary. In particular, we insist that the coordinates $i_{k_l(R)+1}, \ldots, i_{k_l(r)}$ are all equal to $i_l^{\min}$.

Note that we use (2.4) here. We have

$$
\begin{aligned}
\frac{\mu(\pi(Q(\omega,R)))}{\mu(\pi(Q(\omega,r)))} &= \frac{\prod_{l=1}^{d}\left(\prod_{j=0}^{k_l(R)-1} p_l(\sigma^j\omega)\right)}{\prod_{l=1}^{d}\left(\prod_{j=0}^{k_l(r)-1} p_l(\sigma^j\omega)\right)} = \frac{1}{\prod_{l=1}^{d}\left(\prod_{j=k_l(R)}^{k_l(r)-1} p_l(\sigma^j\omega)\right)} \\
&= \prod_{l=1}^{d}\left(\frac{1}{p_l^{\min}}\right)^{k_l(r)-k_l(R)} \\
&\geqslant \prod_{l=1}^{d}\left(\frac{1}{p_l^{\min}}\right)^{-\log r/\log n_l - 1 + \log R/\log n_l} \\
&\geqslant p^d\left(\frac{R}{r}\right)^{s}
\end{aligned}
$$

where $p = \min_l p_l^{\min} > 0$ is a constant as before. Since we can choose a sequence of parameters $x = \pi(\omega)$, $r < R$ satisfying the above with $R/r \to \infty$, it follows that $\overline{\dim}_{\mathrm{reg}}\mu \geqslant s$, completing the proof for the upper regularity dimension.

We proceed in much the same way for the lower regularity dimension, assuming the same preliminary work. For $l = 1, \ldots, d$, choose $p_l^{\max} = \max_{j\in\mathcal{I}} p_l(\mathbf{j})$ and let $i_l^{\max} \in \mathcal{I}$ be an element achieving this maximum. Again, if multiple options exist, any will suffice. Let $t = \sum_{l=1}^{d} \frac{-\log p_l^{\max}}{\log n_l}$ be the target dimension. Starting with the lower bound again uses (2.3). Let $x = \pi(\omega) \in F$ and $0 < r < R \leqslant 1$, assuming without loss of generality that $r < R/n_d$, which guarantees $k_l(R) < k_l(r)$ for all l. Then

$$
\begin{aligned}
\frac{\mu(\pi(Q(\omega,R)))}{\mu(\pi(Q(\omega,r)))} &= \frac{1}{\prod_{l=1}^{d}\left(\prod_{j=k_l(R)}^{k_l(r)-1} p_l(\sigma^j\omega)\right)} \\
&\geqslant \prod_{l=1}^{d}\left(\frac{1}{p_l^{\max}}\right)^{k_l(r)-k_l(R)} \\
&\geqslant \prod_{l=1}^{d}\left(\frac{1}{p_l^{\max}}\right)^{-\log r/\log n_l - 1 + \log R/\log n_l} \\
&\geqslant p^d\left(\frac{R}{r}\right)^{t}
\end{aligned}
$$

with p the constant defined previously. It follows that $\underline{\dim}_{\mathrm{reg}}\mu \geqslant t$.

For the upper bound, the goal is to construct sequences x and $0 < r < R$ for which the expected dimension is attained. Let $0 < r < R$ be any reals satisfying the previous restrictions which ensure that $k_{l+1}(r) < k_l(R) < k_l(r) < k_{l-1}(R)$ holds for all l. The assumption that the n_l are strictly increasing still holds in this setting, so such reals exist. Given these r and R, choose $\omega \in \mathcal{I}^{\mathbb{N}}$ so that

$$\omega = (i_1, \ldots, i_{k_d(R)}, i_d^{\max}, \ldots, i_d^{\max}, i_{k_d(r)+1}, \ldots, i_{k_2(R)},\\ i_2^{\max}, \ldots, i_2^{\max}, i_{k_2(r)+1}, \ldots, i_{k_1(R)}, i_1^{\max}, \ldots, i_1^{\max}, i_{k_1(r)+1}, \ldots).$$

Recall that $i_l^{\max}$ is an element of $\mathcal{I}$ which attains $p_l^{\max}$ and all coordinates not stated as $i_l^{\max}$ are arbitrary. Then, as the $k_l(\cdot)$ are all distinct and well-ordered,

$$\begin{aligned}\frac{\mu(\pi(Q(\omega,R)))}{\mu(\pi(Q(\omega,r)))} &= \frac{1}{\prod_{l=1}^{d}\left(\prod_{j=k_l(R)}^{k_l(r)-1} p_l(\sigma^j\omega)\right)}\\ &= \prod_{l=1}^{d}\left(\frac{1}{p_l^{\max}}\right)^{k_l(r)-k_l(R)}\\ &\leqslant \prod_{l=1}^{d}\left(\frac{1}{p_l^{\max}}\right)^{-\log r/\log n_l+\log R/\log n_l+1}\\ &\leqslant p^{-d}\left(\frac{R}{r}\right)^{t},\end{aligned}$$

with p a constant as before. Due to the construction, a sequence of r and R can be chosen to satisfy $R \to 0$ and $R/r \to \infty$, simultaneously defining a naturally associated sequence of $x = \pi(\omega)$. These sequences satisfy all of the above and thus $\underline{\dim}_{\mathrm{reg}}\mu \leqslant t$, as desired.

2.3.5 Sequences and associated measures

We start by explaining our method and then we specialise to the particular cases of Theorem 20 in subsequent subsections. For convenience, we assume that $x_k \searrow 0$, $x_k - x_{k+1} \searrow 0$ (decreasing gaps) and that $p(k)$ ($k \in \mathbb{N}$) can be extended to a decreasing L^1 function p on the whole of $[0, \infty)$. These conditions are obviously satisfied for the examples we consider. Throughout this thesis we write $f(x) = O(g(x))$ to mean $cg(x) \leqslant f(x) \leqslant Cg(x)$ for constants c, C independent of x.

For $0 < r < 1$ and $x \in \mathrm{supp}(\mu)$, let $\overline{k}(x,r), \underline{k}(x,r)$ be the unique integers such that $x_{\overline{k}(x,r)+1} < x - r \leqslant x_{\overline{k}(x,r)}$ and $x_{\underline{k}(x,r)} \leqslant x + r < x_{\underline{k}(x,r)-1}$ where we adopt the convention that $\overline{k}(x,r) = \infty$ when $x - r \leqslant 0$. Therefore

$$\mu(B(x,r)) = \frac{1}{\sum_{n=1}^{\infty} p(n)} \sum_{i=\underline{k}(x,r)}^{\overline{k}(x,r)} p(i).$$

Of course there is a possibility that $x_{\overline{k}(x,r)} = x$ when $x \neq 0$ and then $\mu(B(x,r)) = \mu(\{x\})$. Thus, given $0 < r < R < 1$, we will consider three different cases:

1. $x \in \mathrm{supp}(\mu) \setminus \{0\}$ such that $x_{\overline{k}(x,r)} = x_{\overline{k}(x,R)} = x$,

2. $x \in \operatorname{supp}(\mu) \setminus \{0\}$ such that $x_{\overline{k}(x,R)} < x_{\overline{k}(x,r)} = x$,

3. $x \in \operatorname{supp}(\mu)$ such that $x_{\overline{k}(x,R)} \leqslant x_{\overline{k}(x,r)} < x$ or $x = 0$.

Case 1 is trivial since

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} = \frac{\mu(\{x\})}{\mu(\{x\})} = 1 \leqslant \left(\frac{R}{r}\right)^0$$

and so we omit further discussion of it. We now consider case 3, which is the most important. Since $p(x)$ is decreasing, we have

$$p(n+1) \leqslant \int_n^{n+1} p(x)dx \leqslant p(n)$$

and therefore

$$\begin{aligned} \frac{1}{\sum_{n=1}^{\infty} p(n)} \int_{\underline{k}(x,r)}^{\overline{k}(x,r)+1} p(x)dx \leqslant \mu(B(x,r)) &= \frac{1}{\sum_{n=1}^{\infty} p(n)} \sum_{i=\underline{k}(x,r)}^{\overline{k}(x,r)} p(i) \\ &\leqslant \frac{1}{\sum_{n=1}^{\infty} p(n)} \int_{\underline{k}(x,r)-1}^{\overline{k}(x,r)} p(x)dx. \end{aligned}$$

Hence for any $0 < r < R < 1$ and any $x \in \operatorname{supp}(\mu)$ in case 3 we have

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant \frac{\int_{\underline{k}(x,R)-1}^{\overline{k}(x,R)} p(x)dx}{\int_{\underline{k}(x,r)}^{\overline{k}(x,r)+1} p(x)dx} \leqslant \frac{\int_{\underline{k}(x,R)-1}^{\overline{k}(x,R)} p(x)dx}{\int_{\underline{k}(x,r)}^{\overline{k}(x,r)} p(x)dx}.$$

To simplify this expression for convenience, we assume that $p(x)$ does not decay faster than exponentially in the sense that there is an $\alpha > 0$ such that $p(n)/p(n+1) \leqslant \alpha$ for all $n \in \mathbb{N}$. (This is satisfied for all $p(n)$ we consider.) Then

$$\begin{aligned} \int_{\underline{k}(x,R)-1}^{\overline{k}(x,R)} p(x)dx &= \int_{\underline{k}(x,R)-1}^{\underline{k}(x,R)} p(x)dx + \int_{\underline{k}(x,R)}^{\overline{k}(x,R)} p(x)dx \\ &\leqslant \alpha^2 \int_{\underline{k}(x,R)}^{\underline{k}(x,R)+1} p(x)dx + \int_{\underline{k}(x,R)}^{\overline{k}(x,R)} p(x)dx \\ &\leqslant (\alpha^2+1) \int_{\underline{k}(x,R)}^{\overline{k}(x,R)} p(x)dx \end{aligned}$$

Thus in case 3 we have

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant (\alpha^2+1) \frac{\int_{\underline{k}(x,R)}^{\overline{k}(x,R)} p(x)dx}{\int_{\underline{k}(x,r)}^{\overline{k}(x,r)} p(x)dx}.$$

In case 2, we get

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant (\alpha^2+1)\frac{\int_{\underline{k}(x,R)}^{\overline{k}(x,R)} p(x)dx}{\mu(\{x\})}.$$

In what follows we will drop the constants (α^2+1) to simplify notation. We can now apply these bounds to specific sequences and measures to get upper bounds for the upper regularity dimension. The lower bounds will be provided by Theorem 3.

Polynomial-polynomial

Let $p(n) = n^{-\omega}$ and $x_n = n^{-\lambda}$ with $\lambda > 0$ and $\omega > 1$. Let $s = \frac{\omega-1}{\lambda}$ be the target dimension and note that

$$(\overline{k}(x,r)+1)^{-\lambda} < x - r \leqslant \overline{k}(x,r)^{-\lambda}$$
$$\underline{k}(x,r)^{-\lambda} \leqslant x + r < (\underline{k}(x,r)-1)^{-\lambda}$$

with $\overline{k}(x,r) = \infty$ if $x - r \leqslant 0$ and thus (for $x > r$)

$$(x-r)^{-1/\lambda} - 1 < \overline{k}(x,r) \leqslant (x-r)^{-1/\lambda}$$
$$(x+r)^{-1/\lambda} \leqslant \underline{k}(x,r) < (x+r)^{-1/\lambda} + 1.$$

We first consider case 3. By our previous calculations, for any $0 < r < R < 1$ and for any $x \in \mathrm{supp}(\mu)$ satisfying the conditions required by case 3, we get (up to constants which we ignore)

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant \frac{\int_{\underline{k}(x,R)}^{\overline{k}(x,R)} x^{-\omega}dx}{\int_{\underline{k}(x,r)}^{\overline{k}(x,r)} x^{-\omega}dx} \leqslant \frac{\overline{k}(x,R)^{1-\omega} - \underline{k}(x,R)^{1-\omega}}{\overline{k}(x,r)^{1-\omega} - \underline{k}(x,r)^{1-\omega}}$$
$$\leqslant \frac{(x+R)^s - \max\{x-R,0\}^s}{(x+r)^s - \max\{x-r,0\}^s}. \tag{2.5}$$

We are interested in the supremum of this upper bound taken over all $x \in [0,1]$. It turns out that this can be controlled from above by a constant multiple of the bound evaluated at $x = 0$ or $x = R$. We demonstrate this by repeated application of Taylor's theorem for $(1+y)^s$ as a function of y close to 0. In particular, there exist constants $\varepsilon \in (0,1)$ and $C > 0$ (depending only on s) such that for any $y \in [-\varepsilon, \varepsilon]$

$$1 + sy - Cy^2 \leqslant (1+y)^s \leqslant 1 + sy + Cy^2,$$

that is $(1+y)^s = 1 + sy + O(y^2)$. We may assume $r < \varepsilon^2 R$ and we consider distinct cases (a), (b) and (c).

(a) Assume $x \in [0,r]$. In this case the upper bound is decreasing so a bound obtained at $x = 0$ will be a bound for the whole region. When $x = 0$ it follows immediately from equation (2.5) that

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant \left(\frac{R}{r}\right)^s.$$

(b) Assume $r < x < R$. If $x > r/\varepsilon$, then using (2.5) gives

$$\begin{aligned}\frac{\mu(B(x,R))}{\mu(B(x,r))} &\leqslant \frac{(2R/x)^s}{(1+r/x)^s - (1-r/x)^s} \leqslant \frac{(2R/x)^s}{1 + sr/x - 1 + sr/x + O((r/x)^2)} \\ &= (2R/x)^s O(x/r) \leqslant O\left(\left(\frac{R}{r}\right)^{\max\{s,1\}}\right).\end{aligned}$$

Similarly, if $x < r/\varepsilon$, then

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant \frac{(2R)^s}{(x+r)^s - (x-r)^s} = O\left(\left(\frac{R}{r}\right)^s\right).$$

The first bound here is controlled from above by the behaviour at $x = 0$ if $s \geqslant 1$ and $x = R$ if $s < 1$ (see below) whilst the second one is simply controlled by the behaviour at $x = 0$.

(c) Assume $x \in [R, 1]$. If $x \leqslant R/\varepsilon$, then by (2.5):

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant \frac{(x+R)^s - (x-R)^s}{(x+r)^s - (x-r)^s} = \frac{(1+R/x)^s - (1-R/x)^s}{(1+r/x)^s - (1-r/x)^s}$$

and so we use Taylor's Theorem to obtain

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant \frac{2^s}{1 + sr/x + O((r/x)^2) - 1 + sr/x + O((r/x)^2)} \leqslant O\left(\frac{x}{r}\right) = O\left(\frac{R}{r}\right).$$

If $x > R/\varepsilon$, then Taylor's theorem can be used on $(1 + R/x)^s$ as well, yielding

$$\begin{aligned}\frac{\mu(B(x,R))}{\mu(B(x,r))} &\leqslant \frac{(1+R/x)^s - (1-R/x)^s}{(1+r/x)^s - (1-r/x)^s} \\ &\leqslant \frac{1 + sR/x - 1 + sR/x + O((R/x)^2)}{1 - sr/x - 1 + sr/x + O((r/x)^2)} = O\left(\frac{R}{r}\right)\end{aligned}$$

as desired. In particular, the bounds attained here are controlled from above by the behaviour at $x = R$. This completes the proof in the original case 3.

We now consider case 2, where we have

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant \frac{(x+R)^s - \max\{x-R, 0\}^s}{x^{\omega/\lambda}}$$

up to a constant which we ignore. This upper bound is decreasing in x and the case 2 assumption forces a lower bound on x in terms of r. Indeed, let $x = n^{-\lambda}$

and note that, since we are in case 2, we have

$$n^{-\lambda} - r > (n+1)^{-\lambda}.$$

It follows that

$$r < n^{-\lambda} - (n+1)^{-\lambda} = \frac{-1}{\lambda} \int_{n+1}^{n} z^{-\lambda-1} dz = \frac{1}{\lambda} n^{-\lambda-1}$$

and rearranging gives

$$x = n^{-\lambda} > (\lambda r)^{\lambda/(\lambda+1)}.$$

Therefore, we have

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant \frac{((\lambda r)^{\lambda/(\lambda+1)} + R)^s - \max\left\{(\lambda r)^{\lambda/(\lambda+1)} - R, 0\right\}^s}{(\lambda r)^{\omega/(\lambda+1)}}$$

and we split into two further subcases according to which term dominates in the numerator.

(i) If $(\lambda r)^{\lambda/(\lambda+1)} < R$, then

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant \frac{(2R)^s}{(\lambda r)^{\omega/(\lambda+1)}} \leqslant O\left((R/r)^s\right)$$

provided $s \geqslant \omega/(\lambda+1)$. If this is not the case, then simple algebra yields $s < 1$. This, combined with our assumption $(\lambda r)^{\lambda/(\lambda+1)} < R$, gives

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant \frac{(2R)^s}{(\lambda r)^{\omega/(\lambda+1)}} \leqslant O\left(R/r\right).$$

(ii) If $(\lambda r)^{\lambda/(\lambda+1)} \geqslant R$, then

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant \frac{((\lambda r)^{\lambda/(\lambda+1)} + R)^s - ((\lambda r)^{\lambda/(\lambda+1)} - R)^s}{(\lambda r)^{\omega/(\lambda+1)}}$$

and applying Taylor series estimates in $R/(\lambda r)^{\lambda/(\lambda+1)} \to 0$ similar to above we obtain

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant \frac{(\lambda r)^{s\lambda/(\lambda+1)}}{(\lambda r)^{\omega/(\lambda+1)}} O\left(R/(\lambda r)^{\lambda/(\lambda+1)}\right) \leqslant O\left(R/r\right).$$

Combining the above estimates gives $\overline{\dim}_{\mathrm{reg}}\mu \leqslant \max\{s, 1\}$. Finally, note that $\overline{\dim}_{\mathrm{loc}}(x,\mu) = 0$ when $x \neq 0$, since such x are atoms. However, when $x = 0$, the above estimates immediately give $\overline{\dim}_{\mathrm{loc}}(0,\mu) = s$ and it is well-known that $\mathrm{supp}(\mu) = 1$, which, combined with Theorem 3, completes the proof of the 'polynomial-polynomial' part of Theorem 20.

Exponential-exponential

Let $x_n = \lambda^n$ with associated probabilities $p(n) = \omega^n$, where $\lambda, \omega \in (0,1)$ and let $s = \frac{\log \omega}{\log \lambda}$ be the target dimension. The situation is much simpler than in the 'polynomial-polynomial' case due to the exponential convergence allowing rougher estimates. In particular, for any $x \in [0,1]$ and $0 < r < R$ with (r/R) sufficiently small, we have

$$\begin{aligned} \frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant \frac{\int_{\underline{k}(x,R)-1}^{\overline{k}(x,R)} \omega^x dx}{\int_{\underline{k}(x,r)}^{\overline{k}(x,r)+1} \omega^x dx} = \frac{\omega^{\underline{k}(x,R)-1} - \omega^{\overline{k}(x,R)}}{\omega^{\underline{k}(x,r)} - \omega^{\overline{k}(x,r)+1}} &\leqslant \frac{\omega^{\underline{k}(x,R)-1}}{\omega^{\underline{k}(x,r)}(1-\omega)} \\ &\leqslant \frac{1}{\omega^2(1-\omega)} \frac{(x+R)^s}{(x+r)^s} \\ &\leqslant O\left((R/r)^s\right) \end{aligned}$$

which proves that $\overline{\dim}_{\mathrm{reg}}\mu \leqslant s$. Again, it is clear that $\overline{\dim}_{\mathrm{loc}}(x,\mu) = 0$ when $x \neq 0$ but $\overline{\dim}_{\mathrm{loc}}(0,\mu) = s$ and it is well-known that $\mathrm{supp}(\mu) = 0$, which resolves the 'exponential-exponential' part of Theorem 20.

Mixed rates

We first consider the case where $x_n = n^{-\lambda}$ with associated probability vector $p(n) = \omega^n$, where $\lambda > 0$ and $\omega \in (0,1)$. Choosing $x = 0$ and $r = R/2$ we get

$$\frac{\mu(B(0,R))}{\mu(B(0,R/2))} \geqslant \frac{\int_{\underline{k}(x,R)}^{\infty} \omega^x dx}{\int_{\underline{k}(x,R/2)-1}^{\infty} \omega^x dx} \geqslant \omega^{R^{-1/\lambda} - (R/2)^{-1/\lambda}} \to \infty$$

as $R \to 0$, which proves that μ is not doubling, as required.

We now consider the opposite case, where $x_n = \lambda^n$ with associated probability vector $p(n) = n^{-\omega}$, where $\lambda \in (0,1)$ and $\omega > 1$. Curiously, and in contrast to the previous case, the measure is 'very doubling' at 0. As such, to demonstrate that the measure is non-doubling, we choose $x = R$ and $r = R/2$. Our previous estimates yield that for sufficiently small $R > 0$ and up to a constant that we ignore

$$\frac{\mu(B(R,R))}{\mu(B(R,R/2))} \geqslant \frac{(-\log 2R)^{1-\omega}}{(-\log \frac{3R}{2})^{1-\omega} - (-\log \frac{R}{2})^{1-\omega}} \to \infty$$

as $R \to 0$, proving that μ is non-doubling.

Chapter 3

Quantifying doubling and uniform perfectness

3.1 Introduction

The goal of this chapter is to quantify how doubling a measure is via the upper regularity dimension and how uniformly perfect it is through the lower regularity dimension. This will start with a simple relationship between these notions which will then help understand a classic technical result regarding the measure of balls. The notion of quasisymmetric invariance will then be studied by calculating bounds for the regularity dimensions of pushforward measures under such maps with respect to the dimensions of the original measures. Finally a relation between the lower regularity dimension and a regularity of measures property recently introduced in Diophantine approximation will be discussed. This leads to a more quantitative statement of a recent theorem in the Diophantine approximation on Kleinian groups.

3.2 Results

We start by establishing some well known facts and introducing our results. Section 3.2.1 will study the link between the regularity dimensions and the doubling constants and constants of uniform perfectness and these notions will help to quantify a well known technical proposition in Section 3.2.2. Quasisymmetric homeomorphisms will be introduced in Section 3.2.3 with the goal of understanding how the regularity dimensions can be changed under the action of such maps. Finally Section 3.2.4 will be spent discussing a result from Diophantine approximation and relating it to our work on the regularity dimensions.

3.2.1 Quantifying doubling and uniform perfectness

The notions of doubling and uniform perfectness are central in this section and are defined in equations (1.2) and (1.4) in Chapter 1. Also recall from the introduction that a measure is doubling if and only if it has finite upper regularity dimension; similarly for uniform perfectness and the lower regularity dimension. Repeatedly, we have stated that this means the regularity dimensions quantify doubling and uniform perfectness without further elaboration. This can be formalised so the upper regularity dimension of a doubling measure can be stated explicitly as a function of the doubling constants; similarly for the lower regularity dimension with respect to the constants of uniform perfectness when this dimension is strictly positive.

Theorem 1 ([How19]). *Let μ be a doubling measure fully supported on a metric space X with optimal doubling constants $C(\theta)$, then*

$$\overline{\dim}_{\mathrm{reg}}\mu = \inf_{\theta>1} \frac{\log C(\theta)}{\log \theta}.$$

Similarly if μ is a uniformly perfect measure with optimal constants of uniform perfectness $K(\theta)$ then

$$\underline{\dim}_{\mathrm{reg}}\mu = \sup_{\theta>1} \frac{\log K(\theta)}{\log \theta}.$$

Unfortunately these formulae are supremums and infimums and so the regularity dimensions are not even always attained for any θ. This means we cannot restate results regarding the doubling constants using solely the upper regularity dimension.

We consider an example which helps illustrates this result. Let F be the middle third Cantor set and μ the self-similar measure fully supported on F with associated probability vector $\{1/4, 3/4\}$, as introduced in the previous chapter. We know the upper regularity dimension is $\log 4/\log 3$ whilst the lower regularity dimension is $\log(4/3)/\log 3$. Start by just considering balls centred on the origin, of radius 3^{-k} for some $k \in \mathbb{N}$ and with $\theta = 3^l$ for any $l \in \mathbb{N}$. Due to the structure of F it is clear that

$$\mu(B(0, 3^{-k})) = 4^l \mu(B(0, 3^{-(k+l)})).$$

Choosing $C(\theta)$ to be optimal means that

$$C(\theta) = \sup\left\{\frac{\mu(B(x, R))}{\mu(B(x, R/\theta))} : x \in \mathrm{supp}(\mu),\ R > 0\right\}.$$

Therefore, in this setting, for $\theta = 3^l$, we have scales $R = 3^{-k}$ and a point $x = 0$

for which the doubling constant is exactly equal to 4^l and so

$$C(3^l) \geqslant 4^l.$$

If this was an equality we would have attained the upper regularity dimension for a finite θ, however the above simple analysis does not cover most points or scales. Heuristically, as the set is self-similar, changing x or r shouldn't change the doubling constant much, but slight perturbations are likely sufficient to increase the doubling constant enough so that the upper regularity dimension is not attained for $\theta = 3^l$. A similar behaviour can be observed at $x = 1$ for the lower regularity dimension. It would be interesting to study this example more and even consider other ranges of θ, perhaps using renewal theory or other techniques.

Question 2. *Is there always a doubling constant with respect to a specific θ which recovers the upper regularity dimension, both for specific cases, such as self-similar measures, and in full generality?*

3.2.2 Quantifying an example of Heinonen

When studying doubling measures a technical proposition is often employed to truly benefit from the regularity of these measures. Simply put, a doubling measure on a uniformly perfect space is also a uniformly perfect measure. This implies the following important bounds. Say μ is a doubling measure on a uniformly perfect, bounded space X, then there exists constants $0 < \lambda_1, \lambda_2 < \infty$ and $0 < t \leqslant s < \infty$ such that for any $x \in X$ and $0 < r$

$$\lambda_1 r^s \leqslant \mu(B(X, r)) \leqslant \lambda_2 r^t.$$

It is not clear where this was first stated, but the standard reference [Hei01] provides this result as an example ([Hei01, Exercise 13.1]) without a proof.

This result opens a number of questions. Notably, how close to zero can the lower regularity dimension be in this setting? Clearly the lower regularity dimension of μ is not independent of the lower dimension of X, since the lower dimension is an upper bound, see [BG98]. But in principle this does not bound the lower regularity dimension from below in any meaningful way.

One could ask if there exists a fixed uniformly perfect space and a sequence of doubling measures on that space which all have the same upper regularity dimension but whose lower regularity dimensions can be made as small as possible. A cursory check of some standard examples, such as self-similar sets and measures, implies this might not be feasible; Theorem 11 of the previous chapter is helpful here. Heuristically, by keeping the upper regularity dimensions of self-similar measures fixed with a goal of decreasing the lower regularity dimension, we must increase a probability from the associated probability vector which is not the minimum of the vector. As this is a probability vector, the sum of the elements is always one so any increase must be accompanied by a decrease in at least one other element. If the lower regularity dimension is to approach zero in the limit a probability must tend towards one and thus all other probabilities must be lowered towards zero. This will inevitably increase the upper regularity dimension after some number of steps, breaking the assumption.

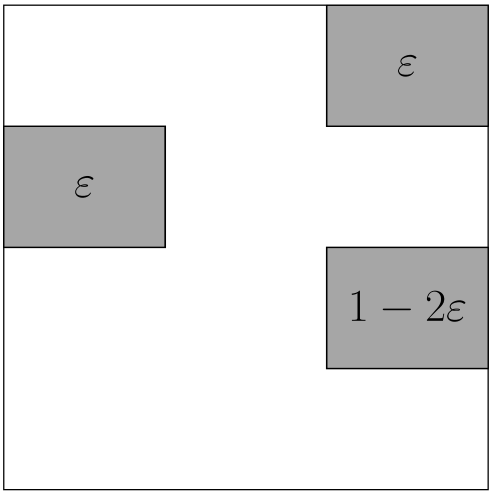


Figure 3: Example of a self-affine set and measure which also demonstrates how the increase in probability of the lower right map will force the remaining probabilities to zero.

This lends credibility to the conjecture that the example of Heinonen implies a lower bound to the lower regularity dimension as a function of the upper regularity dimension. We wish to quantitatively show this result. More precisely, given a measure μ of fixed upper regularity dimension s on a space X of fixed lower dimension l, can we bound the lower regularity dimension of μ as a function of s and l? The following, from [How19], does not quite answer this question as it returns a function of the doubling and uniform perfectness constants. However, as we will discuss, this is closer to the desired solution than it appears.

Proposition 4. *If X is uniformly perfect and μ is a doubling, fully supported measure on X then μ is uniformly perfect. In particular if X is K-uniformly perfect and μ is doubling with doubling constants $C(\theta)$ then*

$$\underline{\dim}_{\mathrm{reg}}\mu \geqslant \frac{\ln(1 - C(8K)^{-1})}{\ln(4K)}.$$

Combining this result with Theorem 1, it follows that a doubling measure on a uniformly perfect space must have lower regularity dimension bounded below by a function of the upper regularity dimension and the constant of uniform perfectness. However, as alluded to previously, Theorem 1 is concerned only with the infimum over all θ and the formula in Proposition 2.1 relies on a specific $C(\theta)$. Thus we cannot, in general, find an exact formula linking the lower regularity dimension and the upper regularity dimension in our setting.

We finish this section with a brief discussion of the sharpness of this result. Assuming positive lower dimension of the space is required here as it is simple to construct spaces of zero lower dimension but finite Assouad dimension. In such a setting there must exist a measure of upper regularity dimension close to the Assouad dimension of the space and so doubling. But any measure on this space cannot be uniformly perfect as the lower regularity dimension is a lower bound to the lower dimension. A trivial such example would be the set of points $\{1/n : n \in \mathbb{N}\}$ with the Euclidean metric. This set is known to have zero lower dimension but full Assouad dimension; doubling measures on this space were explicitly constructed in the previous chapter. On the other hand we will find examples of measures on uniformly perfect spaces which are neither doubling nor uniformly perfect, see Chapter 4 for further information. Thus the obtained bound is justified in demanding both the doubling constants of μ and the constant of uniform perfectness of the set.

There are many examples of uniformly perfect measures, even on doubling spaces, that are not doubling so we cannot interchange the two notions and obtain an analogous result. For instance, one can take a self-similar set with overlaps, this is a doubling space. Numerous uniformly perfect measures exist on such a space and are not doubling, as can be seen in [HT19] where all self-similar measures were shown to be uniformly perfect. Thus a uniformly perfect measure on a space of positive, finite lower and Assouad dimension need not be doubling.

3.2.3 Regularity dimensions under quasisymmetric homeomorphisms

Quasisymmetric homeomorphisms are a generalisation of bi-Lipschitz maps, preserving relative sizes but not necessarily global size. These were first introduced in [BA56, TV80]. In the Euclidean setting quasisymmetric homeomorphisms are equivalent to the often studied quasiconformal homeomorphisms. Recall that $d_X(\cdot,\cdot)$ denotes the metric on the space X; tracking which space we are working in will be important in this section. A homeomorphism $f\colon X \to Y$ is an *η-quasisymmetric homeomorphism (or map)* if there is a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that

$$d_X(x,a) \leqslant t\, d_X(x,b)$$

implies

$$d_Y(f(x), f(a)) \leqslant \eta(t)\, d_Y(f(x), f(b))$$

for all $x, a, b \in X$ and for all $t > 0$.

Equivalently there exists a homeomorphism η as above such that

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leqslant \eta\left(\frac{d_X(x, y)}{d_X(x, z)}\right)$$

for any distinct points $x, y, z \in X$. We will sometimes talk about quasisymmetric homeomorphisms without specifying η, this should be read as there exists a homeomorphism η for which the function is an η-quasisymmetric homeomorphism but the exact form of η is not important at that time and is omitted. Here η is clearly not unique for a given quasisymmetric homeomorphism.

A property of particular interest to us is that doubling and uniform perfection of spaces are quasisymmetric invariants. This can be quantified, so there are bounds on the Assouad and lower dimensions of images of spaces under quasisymmetric embeddings. More precisely, for a set F and a quasisymmetric map f we have

$$\alpha \dim_{\mathrm{A}} F \leqslant \dim_{\mathrm{A}} f(F) \leqslant \frac{1}{\alpha} \dim_{\mathrm{A}} F$$

where α is some constant intrinsic to the map f which will be introduced shortly. This is stated as an exercise in [Hei01] due to Tyson. Note that these estimates are not necessarily true when f is Hölder, see [Luu98]

In fact, there is even a bijection between doubling measures and quasisymmetric homeomorphisms on the real line. To be explicit, every doubling measure μ on $\mathbb{R}$ is associated with a quasisymmetric map $f(x) = \int_0^x d\mu$ and every quasisymmetric $f\colon \mathbb{R} \to \mathbb{R}$ induces a doubling measure $\mu_f(\cdot) = \mathcal{L}(f(\cdot))$ where $\mathcal{L}$ is Lebesgue measure. The classic book [Hei01] contains further details and many other interesting properties of these maps that we will not touch upon in this thesis.

We wish to know if the same holds for doubling and uniformly perfect measures. In particular we will study *pushforward measures* under quasisymmetric homeomorphisms. Given a measure μ on a space X and f a measurable map from X to some space Y, the pushforward measure of μ under f is denoted $f_*\mu$ and is defined by

$$f_*\mu(A) = \mu(f^{-1}(A))$$

for any measurable subset A of Y, where $f^{-1}(A) = \{x \in X \colon f(x) \in A\}$.

To avoid having trivial upper and lower regularity dimensions of μ it is reasonable to assume that X is doubling and uniformly perfect. This then lets us employ the following theorem.

Theorem 5 ([Hei01, Theorem 13.11])**.** *A quasisymmetric homeomorphism f of a uniformly perfect space X is η-quasisymmetric with η of the form*

$$\eta(t) = c_\eta \max\left\{t^\alpha, t^{1/\alpha}\right\},$$

where $c_\eta \geqslant 1$ and $\alpha \in (0,1]$ depend only on f and X.

For clarity we will often write η_α to indicate the homeomorphism η associated with the constant α as described here. Section 3 of [TV80] proves this result and explicitly calculates α.

For such quasisymmetric homeomorphisms we will prove that doubling and uniform perfectness of measures are also invariants, mirroring the geometric setting.

Theorem 6 ([How19])**.** *Let X be a uniformly perfect space and μ be doubling on X. When f is an η_α-quasisymmetric homeomorphism the following bounds hold*

$$\alpha\,\overline{\dim}_{\mathrm{reg}}\mu \leqslant \overline{\dim}_{\mathrm{reg}} f_*\mu \leqslant \overline{\dim}_{\mathrm{reg}}\mu/\alpha$$

and

$$\alpha\,\underline{\dim}_{\mathrm{reg}}\mu \leqslant \underline{\dim}_{\mathrm{reg}} f_*\mu \leqslant \underline{\dim}_{\mathrm{reg}}\mu/\alpha$$

where $f_*\mu = \mu \circ f^{-1}$ is the pushforward of μ.

3.2.4 Uniformly perfect and weakly absolutely α-decaying measures

A property that has appeared recently in Diophantine approximation is the notion of *weakly absolutely α-decaying.* This was first introduced in [BGSV16] following the previous uses of friendly measures by [KLW04] and quasi-decaying measures by [DFSU18, DFSU15]. A measure μ is weakly absolutely α-decaying for some $\alpha > 0$ when there exists constants $C, R_0 > 0$ such that for all $\varepsilon > 0$

$$\mu(B(x, \varepsilon R)) \leqslant C\varepsilon^\alpha \mu(B(x,R))$$

for all $x \in X$ and $R < R_0$.

This property has some resemblance to the ideas of doubling and uniformly perfect measures, in the sense that it compares the measure of a ball to the measure of another ball of same centre but different radius and where the ratio between the two radii is important. The following was shown in [How19].

Proposition 7. *If a measure μ has positive lower regularity dimension then*

$$\underline{\dim}_{\mathrm{reg}}\mu = \sup\left\{\alpha \colon \mu \textit{ is weakly absolutely } \alpha\textit{-decaying}\right\}.$$

This result actually leads to an equivalent but hopefully more applicable statement of Theorem 2 in [BGSV16] regarding Diophantine approximation. We start with a bit of background to motivate this concept.

A well know result of Dirichlet states that for any $x \in \mathbb{R}$ and $N \in \mathbb{N}$, there exists integers $p, q \in \mathbb{Z}$ such that $1 \leqslant q \leqslant N$ and

$$d_{\mathbb{R}}\left(x, \frac{p}{q}\right) \leqslant \frac{1}{qN}.$$

Here and throughout $d_{\mathbb{R}}$ denotes the Euclidean metric on the real line. This result can be thought of as quantifying how dense the rationals are in the reals. A natural extension is to ask what happens when the approximating term $1/qN$ is replaced by another decreasing function $\psi\colon [0,\infty) \to [0,\infty)$. The goal is then to study the size of the points which satisfy the previous inequality with respect to ψ infinitely often. Formally we call such points ψ-well approximable and denote them by

$$W(\psi) = \left\{x \in \mathbb{R}\colon d_{\mathbb{R}}\left(x, \frac{p}{q}\right) \leqslant \psi(q) \text{ for infinitely many } (p,q) \in \mathbb{Z} \times \mathbb{N}\right\}.$$

Note that usually these results are stated in $\mathbb{R}^d$ but for now we restrict our exposition to the real line. The well known Khintchine's Theorem provides the size of $W(\psi)$ in the context of Lebesgue measure.

Theorem 8. *Let $\mathcal{L}$ be Lebesgue measure. Then*

$$\mathcal{L}(W(\psi)) = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} \psi(r) r < \infty, \\ 1 & \text{if } \sum_{r=1}^{\infty} \psi(r) r = \infty. \end{cases}$$

Analogously there has been much work understanding approximation results in non-Euclidean settings. In particular a version of Dirichlet's Theorem was proved by Patterson [Pat76] for finitely generated Kleinian groups acting on the unit disc model of hyperbolic space.

In this section we work in $(d+1)$-dimensional hyperbolic space using the ball model $\mathbb{D}^{d+1} = \left\{z \in \mathbb{R}^{d+1}\colon d_{\mathbb{R}^{d+1}}(\mathbf{0}, z) < 1\right\}$ with the hyperbolic metric $d_{\mathbb{H}}$ derived by $ds = 2|dz|/(1 - |z|^2)$. Here $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^{d+1}$ is just the origin. Consider the group of orientation-preserving Möbius transformations of the unit ball $\mathbb{D}^{d+1}$, denoted Möb$(\mathbb{D}^{d+1})$. Let G be a geometrically finite, discrete subgroup of Möb$(\mathbb{D}^{d+1})$, such groups are called Kleinian groups. A group of Möbius transformations is geometrically finite if it has a fundamental domain with finitely many sides.

We will be interested in studying the limit set of G, denoted Λ, which is the set of limit points in the unit sphere S^d of any orbit of G in $\mathbb{D}^{d+1}$. Non-elementary Kleinian groups have uncountable limit sets, otherwise the limit set contains either 0, 1 or 2 points which does not provide interesting geometric behaviour.

A Möbius transformation is parabolic if and only if it has a unique fixed point in the boundary of hyperbolic space S^d whilst a hyperbolic element will fix exactly two points in S^d. For any $g \in G$ let $L_g = d_{\mathbb{R}^{d+1}}(\mathbf{0}, g'(\mathbf{0}))^{-1}$, where $d_{\mathbb{R}^{d+1}}(\mathbf{0}, g'(\mathbf{0})) = 1 - d_{\mathbb{R}^{d+1}}(\mathbf{0}, g(\mathbf{0}))^2$ is the Euclidean conformal dilation of g at the origin. The distance L_g will tell us how far a group element will move a point in hyperbolic space towards the boundary and is the equivalent of q in the rational setting. A version of Dirichlet's Theorem due to Patterson [Pat76] is then the following.

Theorem 9. *Let G be a non-elementary, geometrically finite Kleinian group without parabolic elements and let $\{\eta, \eta'\}$ be the pair of fixed points of a hyperbolic element of G. Then there is a constant $c > 0$ with the following property: for all $\xi \in \Lambda$, $N > 1$, there exist $y \in \{\eta, \eta'\}$, $g \in G$ so that*

$$d_{\mathbb{R}^{d+1}}(\xi, g(y)) \leqslant \frac{c}{N} \quad \text{and} \quad L_g \leqslant N.$$

As in the Euclidean setting we can study points which are well-approximable, this time with respect to a point $y \in \Lambda$ which will usually be taken as a fixed point of a hyperbolic or parabolic element of G. Formally, for a Kleinian group G, let $\psi\colon [0, \infty) \to [0, \infty)$ be a decreasing function and define

$$W_y(\psi) = \{\xi \in \Lambda \colon d_{\mathbb{R}^{d+1}}(\xi, g(y)) \leqslant \psi(L_g) \text{ for infinitely many } g \in G\}.$$

As $W_y(\psi)$ is a subset of the limit set we cannot use Lebesgue measure to study the size of the well-approximable numbers. The well studied Patterson-Sullivan measure μ_{PS} is the natural answer to this as non-elementary geometrically finite Kleinian groups are known to carry this measure and in this setting it is an atomless conformal ergodic Borel probability measure. An analogue of Khintchine's Theorem in this setting can be shown through the work of [Pat76, Str94, SV95, BDV06]; in its final form the original 1-dimensional version of Khintchine's Theorem can even be recovered. To avoid over complicating this section we omit the full statement of the hyperbolic analogue.

Returning to the classical setting one last time, a theory of Diophantine approximation on manifolds has also been developed. For instance one can ask what is the size of $\mathcal{M} \cap W(\psi)$ for some submanifold $\mathcal{M}$ of $\mathbb{R}^d$. Size here would be the normalised k-dimensional Lebesgue measure on $\mathcal{M}$ where $\mathcal{M}$ is a k-dimensional manifold. The question then becomes which conditions must be imposed on $\mathcal{M}$

so that an analogue of Khintchine's Theorem can be recovered.

Similarly Beresnevich et al. [BGSV16] asked a similar question for subsets of limit sets in the hyperbolic setting, which is the focus of this section. Let K be a subset of the limit set Λ which supports a non-atomic probability measure μ. The desired regularity property of K in this setting is that μ is weakly absolutely α-decaying. The following result can then be shown.

Theorem 10 ([BGSV16, Theorem 2]). *Let G be a nonelementary, geometrically finite Kleinian group and let y be a parabolic fixed point of G, if there are any, and a hyperbolic fixed point otherwise. Fix $\alpha > 0$, and let K be a compact subset of Λ equipped with a weakly absolutely α-decaying measure μ. Then*

$$\mu(K \cap W_y(\psi)) = 0 \quad \text{if} \quad \sum_{r=1}^{\infty} r^{\alpha-1}\psi(r)^{\alpha} < \infty.$$

Recently, Suomala [Suo20] showed that, given a complete metric space X of finite Assouad dimension, for all $0 < s < \dim_{\mathrm{L}} X$, there exists a measure μ with support X such that $\underline{\dim}_{\mathrm{reg}}\mu = s$. Combining this result with Proposition 7 gives the existence of weakly absolutely α-decaying measures for all $0 < \alpha < \dim_{\mathrm{L}} X$ and thus we can deduce a new version of Theorem 10.

Theorem 11. *Assume G and y are as above. Let K be a compact subset of Λ with lower dimension equal to $s > 0$. For any $0 < \alpha < s$, there exists a weakly absolutely α-decaying measure μ on K and*

$$\mu(K \cap W_y(\psi)) = 0 \quad \text{if} \quad \sum_{r=1}^{\infty} r^{\alpha-1}\psi(r)^{\alpha} < \infty.$$

In particular, if $\sum_{r=1}^{\infty} r^{s-1}\psi(r)^{s} < \infty$ then any weakly absolutely α-decaying measure on K is such that $\mu(K \cap W_y(\psi)) = 0$.

An advantage of writing the theorem with respect to the lower dimension of subsets of the limit set is that the lower dimension of limit sets were calculated in [Fra19]. Therefore, given a limit set, we can quickly check if there will be measures that are weakly absolutely α-decaying such that the sum in the theorem converges for the given α. In [Fra19], Fraser also calculated the regularity dimensions of Patterson-Sullivan measures, providing us with explicit measures that could be used in the theorem for suitable subsets, as the upper and lower regularity dimensions of Patterson-Sullivan measures are strictly positive and finite.

Theorem 11 follows from Proposition 7 and Theorem 10. As such we do not provide a proof here, for the interested reader the proof of Theorem 10 in [BGSV16] is very accessible. For further information on Kleinian groups and

Diophantine approximation we refer the reader to [BGSV16, Fra19] and the references therein.

Weakly absolutely decaying measures were the correct measures to consider in the setting of limit sets of Kleinian groups whereas friendly measures were used in the context of subsets of Euclidean space. It would be a natural extension to study the links between friendly measures and the regularity dimensions, especially given that one of the conditions for a measure to be friendly is that a subset of full measure is 'locally doubling', see [KLW04] for the complete definition.

Question 12. *Is there a relation between the regularity dimensions and other notions of regularity of measures used in Diophantine approximation?*

3.3 Proofs

This proof section will be broken into several subsections that are mostly independent of each other but the notation will remain consistent throughout. In Section 3.3.1 we cover the ideas found in Section 3.2.1 relating the regularity dimensions with their associated regularity constants. In Section 3.3.2 we show Proposition 4. Section 3.3.3 is dedicated to the study of quasisymmetric homemorphisms and Theorem 6. Finally in Section 3.3.4 a short proof of Proposition 7 is provided.

3.3.1 Proof of equivalence of doubling and finite upper regularity dimension

We start by proving the link between the upper regularity dimension and the doubling constants. Recall the notion of doubling implies the existence of constants $C(\theta)$; we wish to pick the optimal constant. Given $\theta > 1$ let

$$C(\theta) = \sup\left\{ \frac{\mu(B(x,R))}{\mu(B(x,R/\theta))} \ : \ x \in \operatorname{supp}(\mu), R > 0 \right\} \geqslant 1.$$

Assume that a measure μ on a space X is doubling and let $\theta > 1$. Therefore, for all $x \in \operatorname{supp}(\mu)$ and $R > 0$

$$\frac{\mu(B(x,R))}{\mu(B(x,R/\theta))} \leqslant C(\theta).$$

Fix $0 < r < R$ and define k to be the unique integer such that $R\theta^{-k} > r \geqslant$

$R\theta^{-k-1}$. Then by telescoping we obtain

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} = \frac{\mu(B(x,R))}{\mu(B(x,\theta^{-1}R))} \times \frac{\mu(B(x,\theta^{-1}R))}{\mu(B(x,\theta^{-2}R))} \times \cdots \\ \times \frac{\mu(B(x,\theta^{-k}R))}{\mu(B(x,\theta^{-k-1}R))} \times \frac{\mu(B(x,\theta^{-k-1}R))}{\mu(B(x,r))}.$$

Thus

$$\begin{aligned}\frac{\mu(B(x,R))}{\mu(B(x,r))} &\leqslant C(\theta)^{k+1}\frac{\mu(B(x,\theta^{-k-1}R))}{\mu(B(x,r))} \\ &\leqslant C(\theta)^{\log(R/r)/\log\theta+1} = C(\theta)\left(\frac{R}{r}\right)^{\log C(\theta)/\log\theta}\end{aligned}$$

and hence $\overline{\dim}_{\mathrm{reg}}\mu \leqslant \log C(\theta)/\log\theta < \infty$, giving the upper bound.

To obtain a lower bound on the upper regularity dimension, it suffices to find, for $s = \inf \frac{\log C(\theta)}{\log\theta}$, a sequence of $x \in X$ and $0 < r < R$, with $R/r \to \infty$, such that

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \geqslant \left(\frac{R}{r}\right)^s.$$

From the definition of doubling we know that $\mu(B(x,\theta r)) \leqslant C(\theta)\mu(B(x,r))$ for all x, r, θ. Fixing θ we pick $C(\theta)$ to be reasonably sharp in the sense that there exists at least one pair of x, r such that $\mu(B(x,\theta r)) \geqslant \frac{1}{2}C(\theta)\mu(B(x,r))$. Choosing $C(\theta)$ to be optimal as above would ensure this for instance.

Recall $s = \inf_{\theta>1}\frac{\log C(\theta)}{\log\theta}$. To choose our sequence of x, r and R we simply pick any sequence of increasing θ. Then from our choice of $C(\theta)$, the pair x and r are the pair obtained above. The scale R is then fixed by $R = \theta r$. Finally, due to the choice of s,

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \geqslant \frac{1}{2}C(\theta) \geqslant \frac{1}{2}\theta^s = \frac{1}{2}\left(\frac{R}{r}\right)^s,$$

completing the proof for the upper regularity dimension.

The lower regularity dimension result follows similarly. Recall a measure μ is uniformly perfect if and only if it has positive lower regularity dimension, as discussed in Chapter 1, and thus there exists constants $K(\theta)$ for all $\theta > 1$ such that

$$K(\theta) = \inf\left\{\frac{\mu(B(x,R))}{\mu(B(x,R/\theta))} \;:\; x \in \mathrm{supp}(\mu), R > 0\right\} \geqslant 1.$$

For all $x \in \mathrm{supp}(\mu)$ and $0 < r < R$, fix k to be the unique integer such that

$R\theta^{-k} > r \geqslant R\theta^{-k-1}$. Then by the above argument

$$\begin{aligned}\frac{\mu(B(x,R))}{\mu(B(x,r))} &= \frac{\mu(B(x,R))}{\mu(B(x,\theta^{-1}R))} \times \cdots \times \frac{\mu(B(x,\theta^{-k}R))}{\mu(B(x,r))} \\ &\geqslant K(\theta)^k \frac{\mu(B(x,\theta^{-k}R))}{\mu(B(x,r))} \geqslant K(\theta)^k.\end{aligned}$$

Approximating k yields

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \geqslant K(\theta)^{\log(R/r)/\log(\theta)-1} = K(\theta)^{-1}\left(\frac{R}{r}\right)^{\log K(\theta)/\log\theta}$$

so $\underline{\dim}_{\mathrm{reg}}\mu \geqslant \log K(\theta)/\log\theta$ for any $\theta > 1$ as desired.

For the upper bound, we again find a sequence of $x \in X$ and $0 < r < R$ which will attain the dimension. Recall, for a fixed $\theta > 1$, we can choose x, r such that $\mu(B(x,\theta r)) \leqslant \frac{1}{2}K(\theta)\mu(B(x,r))$.

Let $t = \sup_{\theta>1}\frac{\log K(\theta)}{\log\theta}$ and choose any sequence of strictly increasing $\theta > 1$. This gives us sequences of x, r for which $K(\theta)$ satisfies the above condition and then pick $R = \theta r$. Thus

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant \frac{1}{2}K(\theta) \leqslant \frac{1}{2}\theta^t = \frac{1}{2}\left(\frac{R}{r}\right)^t$$

completing the proof.

3.3.2 Quantifying an example of Heinonen

Let X and μ be as in the statement of Proposition 4 with constant of uniform perfectness K and doubling constants $C(\theta)$. We will rework the proof found in [KLV13, lemma 3.1], paying careful attention to the constants in play. Note there is another proof in [RS18, Lemma 4.5] which could lead to different bounds, but we do not pursue this here.

To start, we are inspired by the proof of a technical result, Proposition B.4.7 in [Gro07]. This states that in our setting there exists a constant $a \in (0,1)$ such that

$$\mu(B(x,aR)) \leqslant (1-a)\mu(B(x,R))$$

for any x, R. We wish to determine a as a function of our known constants and so we prove a particular case of this proposition.

For any $x \in X$ and $R > 0$, as X is uniformly perfect, there exists $y \in X$ such that

$$\frac{R}{2K} \leqslant d(x,y) \leqslant \frac{R}{2}.$$

This choice of y ensures that $B(x,\frac{R}{4K}) \cap B(y,\frac{R}{4K}) = \emptyset$ and $B(x,\frac{R}{4K}) \cup B(y,\frac{R}{4K}) \subseteq$

$B(x,R)$. Thus

$$\begin{aligned}\mu(B(x,R/(4K))) &\leqslant \mu(B(x,R)) - \mu(B(y,R/(4K))) \\ &\leqslant \mu(B(x,R)) - \frac{1}{C(8K)}\mu(B(y,2R)) \\ &\leqslant \mu(B(x,R)) - \frac{1}{C(8K)}\mu(B(x,R)) \\ &= (1 - C(8K)^{-1})\mu(B(x,R)).\end{aligned}$$

Recall $C(8K)$ is the doubling constant of μ where $\theta = 8K$. By iterating this construction we obtain

$$\mu(B(x,R/(4K)^n)) \leqslant (1 - C(8K)^{-1})^n \mu(B(x,R))$$

for any $n \in \mathbb{N}$, as desired.

Returning to the actual question, fix $x \in X$, $0 < r < R$ and choose $n \in \mathbb{N}$ such that $(4K)^{-n-1}R < r \leqslant (4K)^{-n}R$ so that $B(x,r) \subseteq B(x,R/(4K)^n)$. Then

$$\begin{aligned}\frac{\mu(B(x,R))}{\mu(B(x,r))} &\geqslant \frac{\mu(B(x,R))}{(1 - C(8K)^{-1})^n \mu(B(x,R))} \\ &\geqslant (1 - C(8K)^{-1})^{\frac{-\ln(R/r)}{\ln(4K)}+1} \\ &= (1 - C(8K)^{-1}) \left(\frac{R}{r}\right)^{\frac{\ln(1-C(8K)^{-1})}{\ln(4K)}}\end{aligned}$$

as desired.

Note that in the proof of [Gro07] one should use the optimal doubling and uniform perfectness constants to obtain the best bound possible, however the result itself is likely not sharp.

3.3.3 Regularity dimensions under quasisymmetric homeomorphism

Whilst Theorem 5 is the key ingredient in the proof of Theorem 6, the following proposition which can be found in [Hei01] is also required.

Proposition 13 ([Hei01, Proposition 10.6]). *When a quasisymmetric homeomorphism $f\colon X \to Y$ is η-quasisymmetric, its inverse f^{-1} is an η'-quasisymmetric homeomorphism with η' given by $\eta'(t) = 1/\eta^{-1}(1/t)$ for $t > 0$.*

It is then clear that a quasisymmetric homeomorphism f on a uniformly perfect space is associated with a homeomorphism $\eta(t) = c_\eta \max\left\{t^\alpha, t^{1/\alpha}\right\}$ and f^{-1} is also a quasisymmetric homeomorphism associated with the function $1/\eta^{-1}(1/t) \leqslant c_\eta^{1/\alpha} \max\left\{t^\alpha, t^{1/\alpha}\right\}$. Note that the homeomorphism $\eta'(t) = c_\eta^{1/\alpha} \max\left\{t^\alpha, t^{1/\alpha}\right\}$ is not exactly $1/\eta^{-1}(1/t)$, but, as it is an upper bound to the desired function, f^{-1} will be an η'-quasisymmetric homeomorphism.

We start by stating some technical calculations which will be relevant for both the upper and lower regularity dimensions. Let $y \in Y$ and $0 < r < R$. Since Y is uniformly perfect, we can find z_1, z_2 such that $z_1 \in B_Y(y, KR) \setminus B_Y(y, R)$ and $z_2 \in B_Y(y, Kr) \setminus B_Y(y, r)$. Without loss of generality, choose $Kr < R$ so that $d_Y(y, z_1) > d_Y(y, z_2)$, this will be required to use the exact formula for η.

Choose any point $a \in B_Y(y, R)$. From our choice of z_1, it is clear that $d_Y(y, a) \leqslant d_Y(y, z_1)$. Thus, as f is quasisymmetric

$$d_X(f^{-1}(y), f^{-1}(a)) \leqslant \eta(1) d_X(f^{-1}(y), f^{-1}(z_1)),$$

and so

$$f^{-1}(B_Y(y, R)) \subseteq B_X(f^{-1}(y), \eta(1) d_X(f^{-1}(y), f^{-1}(z_1)).$$

Similarly, choosing $a \in B_Y(y, R) \setminus B_Y(y, R/K)$, we have $d_Y(y, a) \geqslant d_Y(y, z_1)/K^2$ and so

$$d_X(f^{-1}(y), f^{-1}(a)) \geqslant \eta^{-1}(K^2) d_X(f^{-1}(y), f^{-1}(z_1)).$$

Hence

$$f^{-1}(B_Y(y, R)) \supseteq B_X(f^{-1}(y), \eta^{-1}(K^2) d_X(f^{-1}(y), f^{-1}(z_1))).$$

Similar statements clearly hold for r with z_2.

We can now prove the upper bound for the upper regularity dimension. For any $\varepsilon > 0$,

$$\begin{aligned}
\frac{\mu(f^{-1}(B_Y(y, R)))}{\mu(f^{-1}(B_Y(y, r)))} &\leqslant \frac{\mu(B_X(f^{-1}(y), \eta(1) d_X(f^{-1}(y), f^{-1}(z_1))))}{\mu(B_X(f^{-1}(y), \eta^{-1}(K^2) d_X(f^{-1}(y), f^{-1}(z_2))))} \\
&\leqslant C_\varepsilon \left(\frac{\eta(1) d_X(f^{-1}(y), f^{-1}(z_1))}{\eta^{-1}(K^2) d_X(f^{-1}(y), f^{-1}(z_2))} \right)^{\overline{\dim}_{\mathrm{reg}} \mu + \varepsilon} \\
&\leqslant C_\varepsilon \left(c_\eta^\alpha \eta(1)/\eta^{-1}(K^2) \right)^{\overline{\dim}_{\mathrm{reg}} \mu + \varepsilon} \left(\frac{d_Y(y, z_1)}{d_Y(y, z_2)} \right)^{(\overline{\dim}_{\mathrm{reg}} \mu + \varepsilon)/\alpha} \\
&\leqslant C_\varepsilon \left(c_\eta^\alpha \eta(1)/\eta^{-1}(K^2) \right)^{\overline{\dim}_{\mathrm{reg}} \mu + \varepsilon} \left(\frac{KR}{r} \right)^{(\overline{\dim}_{\mathrm{reg}} \mu + \varepsilon)/\alpha},
\end{aligned}$$

where C_ε is the constant from the definition of the upper regularity dimension of μ with respect to ε. As ε is arbitrarily chosen this completes the upper bound, that is $\overline{\dim}_{\mathrm{reg}} f_* \mu \leqslant \overline{\dim}_{\mathrm{reg}} \mu / \alpha$.

For the lower bound recall f^{-1} is an η'-quasisymmetric homeomorphism where η' is of the form η_α with the same α as in η. Thus the measure $f_*^{-1}(f_* \mu)$ is the pushforward measure of $f_* \mu$ under the quasisymmetric map f^{-1}. Therefore the above bound also holds for this new measure and so $\overline{\dim}_{\mathrm{reg}} f_*^{-1}(f_* \mu) \leqslant$

$\overline{\dim}_{\mathrm{reg}} f_*\mu/\alpha$. Since f is a homeomorphism this yields $\alpha\,\overline{\dim}_{\mathrm{reg}}\mu \leqslant \overline{\dim}_{\mathrm{reg}} f_*\mu$ as expected.

The lower regularity dimension can be shown in much the same way, we start with the lower bound. Let $\varepsilon > 0$,

$$\begin{aligned}\frac{\mu(f^{-1}(B_Y(y,R)))}{\mu(f^{-1}(B_Y(y,r)))} &\geqslant \frac{\mu(B_X(f^{-1}(y),\eta^{-1}(K^2)d_X(f^{-1}(y),f^{-1}(z_1))))}{\mu(B_X(f^{-1}(y),\eta(1)d_X(f^{-1}(y),f^{-1}(z_2))))}\\ &\geqslant D_\varepsilon\left(\frac{\eta^{-1}(K^2)d_X(f^{-1}(y),f^{-1}(z_1))}{\eta(1)d_X(f^{-1}(y),f^{-1}(z_2))}\right)^{\underline{\dim}_{\mathrm{reg}}\mu-\varepsilon}\\ &\geqslant D_\varepsilon\left(\frac{1}{c_\eta^\alpha}\eta^{-1}(K^2)/\eta(1)\right)^{\underline{\dim}_{\mathrm{reg}}\mu-\varepsilon}\left(\frac{d_Y(y,z_1)}{d_Y(y,z_2)}\right)^{\alpha\,(\underline{\dim}_{\mathrm{reg}}\mu-\varepsilon)}\\ &\geqslant D_\varepsilon\left(\frac{1}{c_\eta^\alpha}\eta^{-1}(K^2)/\eta(1)\right)^{\underline{\dim}_{\mathrm{reg}}\mu-\varepsilon}\left(\frac{R}{Kr}\right)^{\alpha\,(\underline{\dim}_{\mathrm{reg}}\mu-\varepsilon)},\end{aligned}$$

where D_ε is the constant stemming from the lower regularity dimension of μ as a function of ε. Letting ε tend to zero finishes the proof of $\underline{\dim}_{\mathrm{reg}} f_*\mu \geqslant \alpha\,\underline{\dim}_{\mathrm{reg}}\mu$.

For the upper bound, we again argue that $f_*^{-1}(f_*\mu) = \mu$ giving $\underline{\dim}_{\mathrm{reg}} f_*^{-1}(f_*\mu) \geqslant \alpha\,\underline{\dim}_{\mathrm{reg}} f_*^{-1}\mu$. Hence, $\underline{\dim}_{\mathrm{reg}} f_*\mu \leqslant \underline{\dim}_{\mathrm{reg}}\mu/\alpha$.

3.3.4 Uniformly perfect and weakly absolutely α-decaying measures

If μ is weakly absolutely α-decaying then $\mu(B(x,\varepsilon R)) \leqslant C\varepsilon^\alpha \mu(B(x,R))$ for any $x \in X$ and $\varepsilon, R > 0$. Thus

$$\frac{\mu(B(x,R))}{\mu(B(x,\varepsilon R))} \geqslant \frac{1}{C}\left(\frac{R}{\varepsilon R}\right)^\alpha$$

and so $\underline{\dim}_{\mathrm{reg}}\mu \geqslant \alpha$.

For the other direction, assume $\underline{\dim}_{\mathrm{reg}}\mu = t$. Then for any $\delta > 0$, there exists $C' > 0$ such that for all $x \in X$ and $R > r > 0$

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \geqslant C'\left(\frac{R}{r}\right)^{t-\delta}.$$

Given $\varepsilon \in (0,1)$ and $R > 0$, choose $r = \varepsilon R$. Inserting this value of r into the above yields

$$\mu(B(x,R)) \geqslant C'\left(\frac{R}{\varepsilon R}\right)^{t-\delta}\mu(B(x,\varepsilon R)).$$

Hence

$$\mu(B(x,\varepsilon R)) \leqslant \frac{1}{C'}\varepsilon^{t-\delta}\mu(B(x,R))$$

and so μ is $(t-\delta)$-decaying.

Chapter 4

Graphs of Brownian motion

4.1 Introduction

Studying the dimension of various random processes has been of interest for quite some time, from [Tay53] to [FLV18]. In this chapter we will consider the Assouad and lower dimensions of graphs of certain random processes, notably Lévy processes and functions defined by stochastic integrals. Then the regularity dimensions of pushforward measures onto graphs of Lévy processes will be computed for doubling measures defined on the unit interval. In particular we will see that pushforwards of doubling maps in this setting are almost surely *not* doubling. This contrasts with the previous chapter's results on quasisymmetric homeomorphisms which conserved doubling; similarly for uniform perfectness. Furthermore, this will provide a nice example of a uniformly perfect and doubling space with a non uniformly perfect and non doubling measure fully supported on that same space.

4.2 Graphs of Lévy processes

Lévy processes $X(t)$ were first introduced by Paul Lévy in 1934 [Lév34] and are defined to be the stochastic processes satisfying:

1 : $X(0) = 0$ almost surely.

2 : For all $t, h > 0$, $X(t+h) - X(t)$ is equal to $X(h)$ in distribution (stationary increments).

3 : For all $0 < t_1 < t_2 < ... < t_k$ the random variables $X(t_i) - X(t_{i-1})$ are independent (independence of increments).

4 : For all $t > 0$ $\lim_{h \to 0} X(t+h) - X(t) = 0$ in probability (continuity).

We can construct $X(t)$ such that it is almost surely right continuous with left limits (denoted càdlàg). Such processes are standard tools in many areas of modern mathematics and its applications. A common example of a Lévy process is the *Wiener process* (or Brownian motion) where property 2 (stationary increments) is replaced by Gaussian increments, so $X(t+h) - X(t)$ is normally distributed with mean 0 and variance h. One can similarly define d-dimensional Brownian motion by considering the vector-valued stochastic process $(W_1, \ldots, W_d)$ where the W_i are independent Wiener processes.

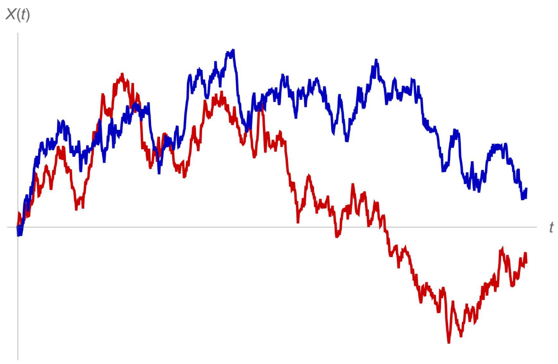


Figure 1: Graph of 2 realisations of one-dimensional Brownian motion.

The geometric properties, such as dimension, of Wiener processes have been a particularly well-studied area. This includes studying the *graphs*, *level sets* and *trails* of such processes which can be thought of as fractals as they often display some *statistical self-affinity*. For any right continuous function $X : \mathbb{R} \to \mathbb{R}$, we define the graph of the function restricted to the interval $I \subseteq \mathbb{R}$ by

$$G_X^I = \{(t, y) \colon\ y = X(t), t \in I\} \cup J,$$

where J is the union of vertical segments joining the discontinuities. The collection of lines J is well defined because X is right continuous. It is clear that if X is continuous then J is empty and as Wiener processes are almost surely continuous J will not matter in this setting and can be safely ignored. This is not true for other Lévy processes and J will actually be needed in our proofs so we include it for generality.

Taylor [Tay53] first calculated the Hausdorff dimension of d-dimensional Brownian motion $B_d : \mathbb{R} \to \mathbb{R}^d$ where he showed that almost surely

$$\dim_{\mathrm{H}} G_{B_1}^{[0,1]} = \dim_{\mathrm{B}} G_{B_1}^{[0,1]} = \frac{3}{2}$$

and for any $d \geqslant 2$

$$\dim_{\mathrm{H}} B_d([0,1]) = \dim_{\mathrm{B}} B_d([0,1]) = 2.$$

A standard survey in this area is [Tay86] which contains many further calculations and references.

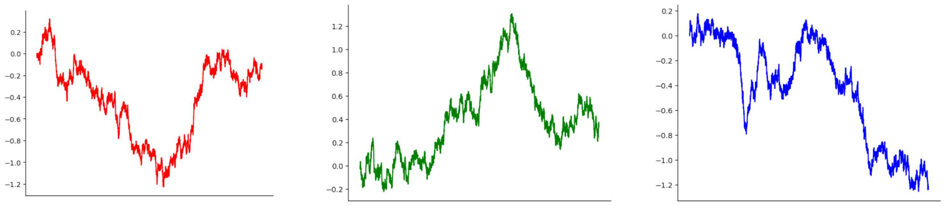


Figure 2: Graphs of 3 realisations of a Brownian motion, a 2-stable Lévy process.

One can formalise the aforementioned statistical self-affinity of Lévy process. However, not all processes have this property so we will restrict to what are called *stable* or *α-stable processes*, that is where, for some $\alpha > 0$,

$$c^{-1/\alpha} X(ct) =^d X(t)$$

for all $c, t > 0$ and $=^d$ means equal in distribution. Two such random processes X, Y are equal in distribution if for all $k \in \mathbb{N}$ and $t_1 < t_2 < \cdots < t_k$, the joint distribution of the random vectors $(X(t_1), \ldots, X(t_k))$ and $(Y(t_1), \ldots, Y(t_k))$ are equal. We will sometimes refer to α as the scaling coefficient of X. The Wiener process is a particular example of a Lévy process which is α-stable with $\alpha = 2$.

It has been shown by [BG62] that such α-stable Lévy processes have

$$\dim_{\mathrm{H}} G_{X_1}^{[0,1]} = \dim_{\mathrm{B}} G_{X_1}^{[0,1]} = \max\{1, 2 - 1/\alpha\},$$

recovering the Brownian motion result above by setting $\alpha = 2$. Note that when $\alpha < 1$ then the graph simply has Hausdorff dimension one with full probability, the lowest possible dimension, as these processes are constant except between certain jump discontinuities. In particular the occurence of these jumps is low enough to not raise this dimension.

Our final condition for these processes is a simple assumption that the distribution $X(1)$ is non-trivial on $\mathbb{R}$. Non-zero on an interval around zero would also work, this is just to ensure the graphs are not just multiple flat lines, such as in a Poisson process. This ensures genuinely interesting geometric behaviour.

Another generalisation of Brownian motion is *fractional Brownian motion*, first introduced by Mandelbrot and van Ness [MvN68]. Index-h fractional Brownian motion (fBm) on $\mathbb{R}$ with $0 < h < 1$ is defined to be the stochastic integral

$$B_h(t) = c(h)^{-1} \int_{-\infty}^{\infty} \left((t - x)_+^{h-1/2} - (-x)_+^{h-1/2} \right) dW(x)$$

where the integral is an Itô integral with respect to the Wiener process, $(x)_+ =$

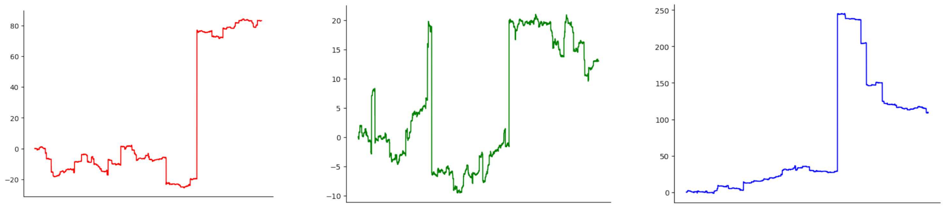


Figure 3: Graphs of 3 realisations of a Lévy process whose increments are Cauchy distributed, a 1-stable Lévy process.

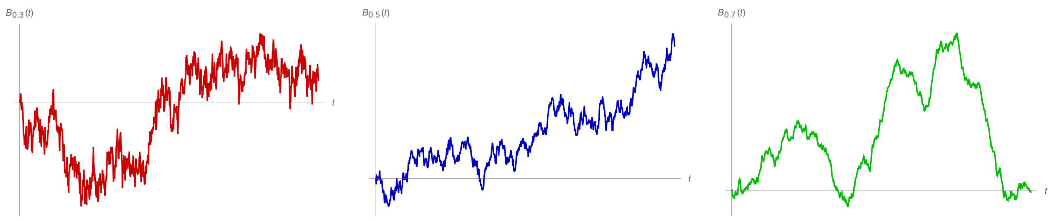
$\max\{0, x\}$ and $c(h) = \Gamma(h + 1/2)$, where Γ denotes the gamma function. Many standard books on stochastic calculus will introduce the Itô intergral, we will later refer to [Pro05] for further properties of this intergal. Equivalently this is a Gaussian random process $B_h(t)$ with:

1 : $B_h(0) = 0$ almost surely.

2 : For all $t, u > 0$, $B_h(t + u) - B_h(t)$ has normal distribution with mean 0 and variance u^{2h} (Gaussian increments).

3 : For all $t > 0$, $\lim_{u\to 0} B_h(t + u) - B_h(t) = 0$ in probability (continuity).

One can see that when $h = 1/2$, $B_{1/2}(t) = B(t)$ is simply Brownian motion. Several equivalent definitions for fBm exist, we choose to use the Itô integral form as our results follow more directly in this setting. Note that fractional Brownian motion is not a Lévy process as its increments are not independent, except when $h = 1/2$ where we recover regular Brownian motion. Much progress has been made on the properties of fBm, see for instance [Adl77, Kah85, Fal15]. Notably it was shown that almost surely, the graph over the unit interval of index-h fBm has Hausdorff dimension $2 - h$.



(a) Graph of fBm with h=0.3. (b) Graph of fBm with h=0.5. (c) Graph of fBm with h=0.7.

The Assouad and lower dimensions have not even been studied in the classic Wiener process setting. For generality, we wish to calculate the Assouad dimension of the graph $G_X^{[0,1]}$ for α-stable Lévy processes and stochastic integrals X. As the Assouad dimension provides information on the extremal local scaling of a set, in this setting it will tell us about the maximal fluctuations of a random process and so we expect the dimension to be full. Similarly for the lower dimension, we expect it to pick up on the sparsest behaviour; in this case the lower dimension should treat the graph as if it were a smooth line.

Theorem 5 ([HY17]). *Let X be an α-stable Lévy process with $\alpha \geqslant 1$ such that $X(1)$ is a random variable whose distribution function is non-vanishing almost everywhere. Then almost surely*

$$\dim_{\mathrm{A}} G_X^{[0,1]} = 2$$

and

$$\dim_{\mathrm{L}} G_X^{[0,1]} = 1.$$

Interestingly the Assouad and lower dimensions of the graph do not depend on the scaling coefficient, unlike the Hausdorff dimension. However we do not obtain a formula for $\alpha \in (0,1)$ for the Assouad dimension as our technique does not extend to this setting; it is possible that with such a parameter, the Assouad dimension would decrease to one as a function of α. But it is also possible that the Assouad dimension of these graphs is simply one, as the distribution of a jump discontinuity of height greater than R in an interval of width r is Poisson with mean r/R^{α}. Further study of this question could result in a novel answer.

Question 6. *For any $\alpha \in (0,1)$, it is known that the graph of an $\alpha-$stable Lévy process almost surely has dimension 1 for many notions of dimension. What is the Assouad dimension of these graphs?*

A particular case of this theorem is, as the Wiener process $W(t)$ is a 2-stable Lévy process, the following.

Corollary 7. *Almost surely*

$$\dim_{\mathrm{A}} G_W^{[0,1]} = 2$$

and

$$\dim_{\mathrm{L}} G_W^{[0,1]} = 1.$$

Brownian motion also provides examples of Salem sets that can have different Hausdorff and Assouad dimensions, we refer the reader to [Kah85] for further discussion on the links between random processes and Salem sets.

Ville Suomala and Changhao Chen, in a personal communication, kindly remarked that the Assouad part of this corollary follows from the graph of Brownian motion having full lower porosity dimension. This approach is inspired by [CG91], where it was shown that the graph of Brownian motion has full upper porosity dimension. However this porosity dimension technique does not extend to our following, more general result.

We can say more about the Assouad dimension of random processes which are functions defined as stochastic integrals.

Theorem 8 ([HY17])**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is zero only finitely often and is C^2 on some interval. Then we define $B_f(t)$ as the function defined by the stochastic integral*

$$B_f(t) = \int_0^t f(x) dW(x).$$

We have that almost surely:

$$\dim_{\mathrm{A}} G_{B_f}^{[0,1]} = 2.$$

In particular, graphs of fractional Brownian motions with indices $0 < h < 1$ have full Assouad dimension almost surely. The conditions in this theorem are designed to ensure f is a reasonably smooth, non-zero function which will allow the random Wiener process to dictate the behaviour of the graph. Usually these integrals are with respect to a random function, but controlling the interaction between a random function and the Wiener process raises issues in this setting and so is left as an open problem.

Question 9. *Can a lower dimension analogue of Theorem 8 be found? Given that this result mirrors the Assouad dimension of the graphs of Brownian motion it is natural to conjecture that the lower dimension in this setting will be almost surely 1.*

The proof of Theorem 5 can also be extended to higher dimensions. Thus one could show the following result.

Let $B_d(t)$ be the d-dimensional Brownian motion, from $\mathbb{R}$ to $\mathbb{R}^d$. Then almost surely

$$\dim_{\mathrm{A}} B_d([0,1]) = d$$

and

$$\dim_{\mathrm{L}} B_d([0,1]) = 1.$$

We can compare this result to the well known

$$\dim_{\mathrm{H}} B_d([0,1]) = 2$$

for $d \geqslant 2$. The Hausdorff dimension being 2 here can be thought as a reflection that higher dimensional Brownian motion is transient, that is almost surely the process will diverge to infinity without returning to a neighbourhood infinitely often, whilst the Assouad dimension shows that there are still areas of maximal fluctuation.

4.3 Proofs regarding the dimensions of graphs

This section will be dedicated to the proofs of Theorems 5 and 8. Section 4.3.4 will provide an outline of a proof for the higher dimensional analogue of Theorem 5.

4.3.1 Assouad and lower dimensions of graphs

In this section we will state a convenient condition to check whether a graph of a function $f : [0,1] \to \mathbb{R}$ has full Assouad dimension or lower dimension one. This will be leveraged in the following sections by showing graphs of different random processes satisfy the conditions stated here.

Let $R_1, R_2 > 0$ be positive numbers and $n > 0$ be an integer. Given a point $a \in \mathbb{R}$, we define $W_n^{R_1 \times R_2}((a, f(a)))$ as the following collection of sets

$$\left\{ \left[a + \frac{i}{n}R_1, a + \frac{i+1}{n}R_1\right] \times \left[f(a) + \frac{j}{n}R_2, f(a) + \frac{j+1}{n}R_2\right] : \right.$$
$$\left. i, j \in \{0, \ldots, n-1\} \right\}.$$

We see that $W_n^{R_1 \times R_2}((a, f(a)))$ is the collection of rectangles of side lengths R_1/n by R_2/n and with disjoint interiors, which partitions the R_1 by R_2 rectangle of lower left vertex $(a, f(a))$.

Then let $N_n^{R_1 \times R_2}((a, f(a)), G_f^{[0,1]}) = \# \left\{ W_n^{R_1 \times R_2}((a, f(a))) \cap G_f^{[0,1]} \right\}$, where $\#$ is the cardinality of the set, this provides the number of rectangles of side lengths R_1/n by R_2/n which intersect the graph within the larger rectangle $[a_i, a_i + R_1] \times [f(a_i), f(a_i) + R_2]$.

The following proposition follows directly from the definition of Assouad dimension and the proof simply relies on noticing the relation between $N_{n_1}^{R_1 \times R_1}((a, f(a)), G_f^{[0,1]})$ and the covering number of an associated ball.

Proposition 10. *If there exists a constant $C > 0$ and sequences:*

$$a_i \in \mathbb{R}, R_i \in (0,1),\ n_i \in \mathbb{N} \quad (\forall i \in \mathbb{N})$$

with $n_i \to \infty$ such that for all $i \in \mathbb{N}$

$$N_{n_i}^{R_i \times R_i}((a_i, f(a_i)), G_f^{[0,1]}) \geqslant Cn_i^2.$$

Then

$$\dim_{\mathrm{A}} G_f^{[0,1]} = 2.$$

Whilst this might seem like a restrictive condition to ask a general function to satisfy, it is quite natural in the setting of Lévy processes due to the almost sure unbounded variation and stable property of the process. Considering squares instead of balls in the definition of Assouad dimension is similar to the definition of the Furstenberg star dimension, which is in fact equivalent to the Assouad dimension, see [CWW16].

Note that one could replace the inequality $N_{n_i}^{R_i \times R_i}((a_i, f(a_i)), G_f^{[0,1]}) \geqslant Cn_i^2$ with the following equality

$$N_{n}^{R_i \times R_i}((a_i, f(a_i)), G_f^{[0,1]}) = n_i^2.$$

This follows from [FY16, Theorem 2.4], where it is shown that a set has full Assouad dimension if and only if it has the unit ball as a weak tangent. This means that any cover of our set is also a cover of a ball and so all smaller squares are needed in the cover. This fact should be clear at the end of the proof in the next section but we leave the inequality in the proposition as it is easier to see the relation with the Assouad dimension in this form.

A similar statement can be made about the lower dimension and again, the proof relies on observing the relation between covering numbers of squares and balls. It is perhaps worth noting that even though the lower dimension is not monotone, as long as the function f is càdlàg then the dimension of the graph with adjoining lines J must be bounded below by 1.

Proposition 11. *If there exists a constant $C' > 0$ and sequences:*

$$a_i \in \mathbb{R}, R_i \in (0,1), \, n_i \in \mathbb{N} \quad (\forall i \in \mathbb{N})$$

with $n_i \to \infty$ such that for all $i \in \mathbb{N}$

$$N_{n_i}^{R_i \times R_i}((a_i, f(a_i)), G_f^{[0,1]}) \leqslant C' n_i$$

and the collection of all hyper-rectangles from the family $W_{n_i}^{R_i \times R_i}((a_i, f(a_i)))$ that *intersect the graph $G_f^{[0,1]}$ includes at least all of the squares in the column*

$$\left\{ \left[a_i + \frac{j}{n_i}R_i, a_i + \frac{j+1}{n_i}R_i\right] \times \left[f(a_i) + \frac{k}{n_i}R_i, f(a_i) + \frac{k+1}{n_i}R_i\right] : \right.$$
$$\left. j = \lfloor n_i/2 \rfloor \text{ and } k = 0, 1, \ldots, n_i - 1 \right\}.$$

Then

$$\dim_{\mathrm{L}} G_f^{[0,1]} = 1.$$

Here $\lfloor \cdot \rfloor$ is the floor function. When n_i is odd, the chosen column is clearly the centre of the square; when n_i is even we simply take the column to the left of centre. In Proposition 10 we pass from the squares considered to balls of same centres with radius similar to the side lengths of the squares. As we are just trying to find a lower bound to the covering number, it suffices to cover one quarter of the ball, which is given by the cover of the square. For the lower dimension the goal is to obtain an upper bound so the entire ball must be understood, thus we must study a ball centred inside the square given by the assumptions. This leads us to ask for intersection in a centred column to guarantee the existence of a point which can be taken as the centre of a ball. The above proposition accounts for this technical point but could be weakened easily; this form suffices for our proof in the next section.

As with the Assouad dimension, the inequality $N_{n_i}^{R_i \times R_i}((a_i, f(a_i)), G_f^{[0,1]}) \leqslant C'n_i$ will actually be an equality in the proof for all i with C' independent of i; however this is not guaranteed as it was previously. In [FHKY19] it was show that a set has zero lower dimension if and only if it has a singleton as a weak tangent but this does not directly imply that a set of lower dimension one must have an interval as a weak tangent. In fact there exist examples of self-similar sets of dimension one with no interval as weak tangents. Whilst a somewhat artificial argument, this behaviour is expected to be the norm.

4.3.2 Graphs of Lévy processes

Let $X(t)$ be an α-stable Lévy process with $\alpha \geqslant 1$. We assume that $X(1)$ is non-vanishing almost everywhere on $\mathbb{R}$ as a random variable, that is, the distribution function of $X(1)$ is 0 only on a set of zero measure. We start by proving the Assouad dimension result of Theorem 5; the lower version will be a simple modification. This proof will aim to construct infinitely many, independent events of positive probability such that the sum of their probabilities is infinite. Then, by Borel-Cantelli, infinitely many of these events must happen. If we have constructed the events correctly this will provide a sequence of rectangles which satisfy Proposition 10, completing the proof.

The events to consider will be stated explicitly later. However, they will be closely related to the events '$N_n^{1\times 1}((0,0), G_X^{[0,1]}) = n^2$' for some $n \in \mathbb{N}$, this is when the graph intersects all parts of the unit square. The probability of such an event can be computed and is a strictly positive number depending only on n; we denote this number by

$$P(n) = P\left(\text{`}N_n^{1\times 1}((0,0), G_X^{[0,1]}) = n^2\text{'}\right).$$

Given a realisation of X, its associated graph $G_X^{[0,1]}$ and a closed rectangle, the graph either intersects the rectangle or it does not. The event '$G_X^{[0,1]}$ hits a specific closed rectangle', as the collection of all outcomes where the graph intersects the rectangle, is measurable. Therefore the event '$N_n^{1\times 1}((0,0), G_X^{[0,1]}) = n^2$' is measurable as the union of finitely many measurable events.

Set $D(j) = [j/n, (j+1)/n]$ for all $j \in \{0, \ldots, n-1\}$ then we can see that

$$P(n) \geqslant P\left(\forall\, k \in \left[1, n^2\right], k \in \mathbb{N}, X\left(\frac{k}{n^2}\right) \in D\left(\left\{\frac{k}{n}\right\} n\right)\right) > 0.$$

Here $\{\cdot\}$ denotes the fractional part function. This follows from our assumption that $X(1)$ is non-trivial almost everywhere on $\mathbb{R}$ and the independent increments property of Lévy processes. It is clear that this restriction could be relaxed to positive on some interval without much effort. To continue, we only require that $P(n)$ is strictly positive so do not explicitly compute it.

The question is now how to obtain infinitely many events. For an α-stable random process X, we can decompose the graph into countably many disjoint parts

$$\bigcup_{i=0}^{\infty} G_X^{I_i} \subseteq G_X^{[0,1]},$$

where I_i are closed intervals with disjoint interiors such that their union is a subset of the unit interval. For our case one could think of this as partitioning the unit interval by intervals of length $1/2^i$. For example take $a_1 = 0$ and for all $i \geqslant 1$ let $a_{i+1} = a_i + 1/2^i$ and $I_i = [a_i, a_i + 1/2^i]$.

Denote by R_i the length of the interval $I_i = [a_i, b_i]$. Since we can take X as a right continuous function, $X(a_i) \in \mathbb{R}$ is defined for all i. For each i we can apply a linear map $T_i : G_X^{I_i} \to [0,1]^2$

$$T_i(x, y) = \left(\frac{1}{R_i}(x - a_i), \frac{1}{R_i^{1/\alpha}}\,(y - X(a_i))\right).$$

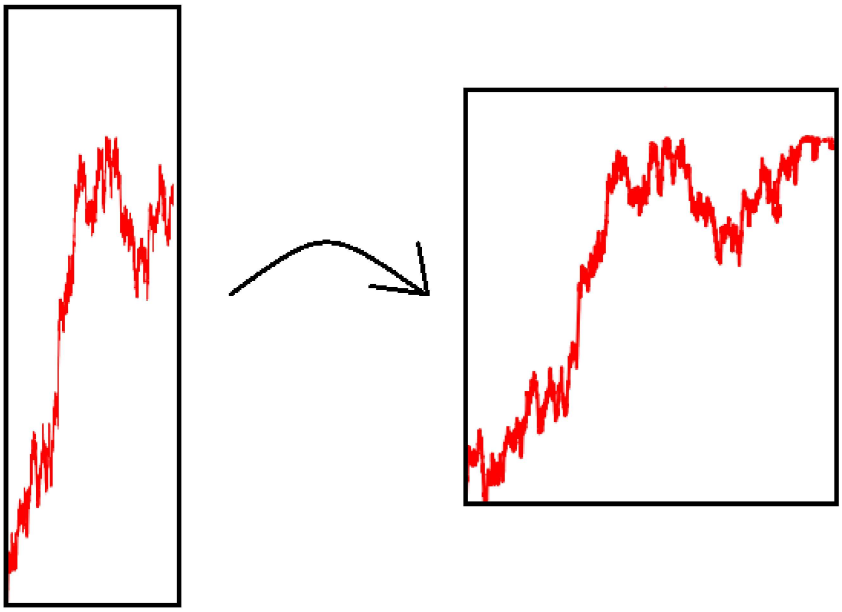


Figure 12: Example of the self-affine map used here.

We wish to study the image of $G_X^{I_i}$ under these maps so identify $T_i(G_X^{I_i}) = G_{X_i}^{[0,1]}$ where X_i is the distribution of the new graph. Due to the scaling behaviour of α-stable processes,

$$X(t) = \frac{X(R_i t)}{R_i^{1/\alpha}} = X_i(t)$$

in distribution and so all the variables X_i are independent α-stable Lévy processes with the same, original distribution X. Geometrically we are mapping tall, thin rectangles to the unit square which is why this scaling is often called self-affinity, see Figure 12 for an illustration.

Let n_i be any sequence of increasing, positive integers such that $\lim_{i\to\infty} n_i = \infty$. Denote by A_i the event '$N_{n_i}^{1\times 1}((0,0), G_{X_i}^{[0,1]}) = n_i^2$'. According to the discussions above, as X_i is equal to X in distribution, we see that the probability of A_i is $P(n_i)$ and so it exists and is strictly positive for each i. Figure 13 displays such an example event.

We can choose n_i to grow slowly enough such that $\sum_i P(n_i) = \infty$, for instance by repeating n_i as many times as desired for any given i. Note that the I_i can be chosen so that the squares are pairwise disjoint and as Lévy processes are Markov, the events A_i are all independent. Then by the (second) Borel-Cantelli Lemma we see that with probability 1, infinitely many events A_i occur. Now if A_i happens, then

$$N_{n_i}^{1\times 1}\left((0,0), G_{X_i}^{[0,1]}\right) = n_i^2.$$

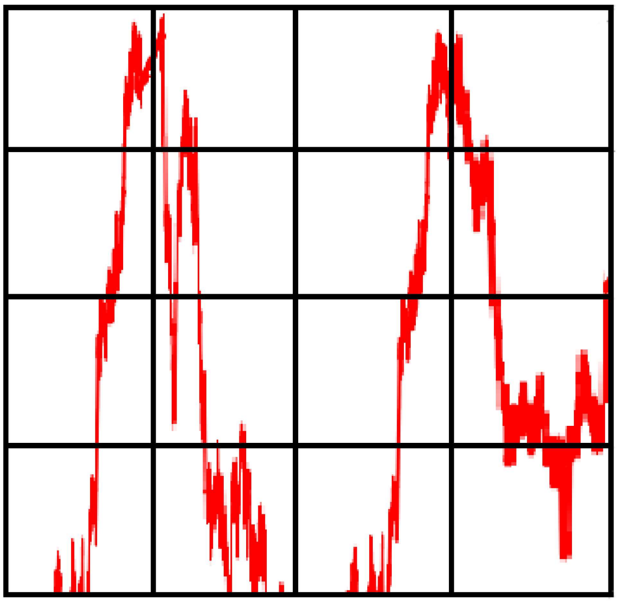


Figure 13: Example event satisfying the above conditions.

Applying the function T_i^{-1} to the graph, we see that

$$N_{n_i}^{R_i\times R_i^{1/\alpha}}\left((a_i,X(a_i)),G_X^{I_i}\right)=n_i^2.$$

Note this last line tells us the number of rectangles, not squares, of side lengths R_i/n_i by $R_i^{1/\alpha}/n_i$ required to cover $G_X^{[0,1]}$ inside the rectangle $[a_i,a_i+R_i]\times[X(a_i),X(a_i)+R_i^{1/\alpha}]$.

Recall $\alpha\geqslant 1$ so $R_i\leqslant R_i^{1/\alpha}$ and all of these rectangles are taller than they are wide. When dividing each of these covering rectangles into squares of side lengths R_i/n_i we see that they must be placed vertically above one another, rather than horizontally and so each of the squares must also intersect the graph as the process is right continuous and we include the lines J. Thus a cover of the graph inside the rectangle $[a_i,a_i+R_i]\times[X(a_i),X(a_i)+R_i^{1/\alpha}]$ by squares of side lengths R_i/n_i requires all possible squares.

Hence, given an event A_i occurs, when restricting attention to the square $[a_i,a_i+R_i]\times[X(a_i),X(a_i)+R_i]$ and considering the covering number by squares of side lengths R_i/n_i we see

$$N_{n_i}^{R_i\times R_i}\left((a_i,X(a_i)),G_X^{I_i}\right)=n_i^2.$$

The events '$N_{n_i}^{R_i\times R_i}\left((a_i,X(a_i)),G_X^{I_i}\right)=n_i^2$' are the ones that we wish to use in Proposition 10.

As infinitely many events A_i occur, using Proposition 10, we see that almost surely

$$\dim_{\mathrm{A}} G_X^{[0,1]} = 2.$$

It is worth noting here that the events used in this proof can be made more specific. In particular, within a given column $[a + \frac{j}{n}R, a + \frac{j+1}{n}R] \times [X(a), X(a) + R^{1/\alpha}]$ with a, R and n as before and for $j \in \{0, 1, \ldots, n-1\}$, the graph $G_X^{[0,1]}$ must, in total, increase or decrease by $R^{1/\alpha}$ to satisfy the events used here. This fluctuation can occur multiple times in any given column. However, the choice of events can be made more precise by asking for the graph to only increase by at least $R^{1/\alpha}$ within the columns with j even and only decrease by the same amount in the odd ones. For instance the graph can satisfy

$$X(a + \frac{j+1}{n}R) - X(a + \frac{j}{n}R) > R^{1/\alpha}$$

and

$$\min_{x,y \in G_X^{[a+\frac{j}{n}R, a+\frac{j+1}{n}R]}, x<y} y - x > \frac{-R^{1/\alpha}}{n}$$

whenever j is even and vice versa for j odd. Geometrically this says that the graph will essentially only increase in even columns and decrease otherwise. This leads to a 'zigzag' behaviour which will be used in the next section.

For the lower dimension we start by modifying our events. Previously, the desired outcome was that all possible squares were needed in the cover, now we ask that only the middle column intersect the graph. That is, for $n \in \mathbb{N}$,

$$G_X^{[0,1]} \cap (0,1]^2 \subseteq \left[\frac{\lfloor n/2 \rfloor}{n}, \frac{\lfloor n/2 \rfloor + 1}{n}\right] \times [0,1]$$

and $N_n^{1\times 1}((0,0), G_X^{[0,1]}) = n + 1$. We recall the discussion in the previous section as to why the middle column is taken here. One could ask for even less intersection, however this method suffices and is more geometrically clear, see Figure 14. As before, this event is measurable and has positive probability given our assumptions.

We then proceed in the same way: partition the graph into countably many parts $\{I_i\}$ of lengths R_i and note that the images of the graph under the transformations T_i still have underlying distribution $X_i =^d X$ as X is α-stable. By choosing a sequence of n_i (of increasing, positive integers) correctly we will have $\sum_i P(n_i) = \infty$ and can denote by A_i the event

$$G_{X_i}^{[0,1]} \cap (0,1]^2 \subseteq \left[\frac{\lfloor n_i/2 \rfloor}{n_i}, \frac{\lfloor n_i/2 \rfloor + 1}{n_i}\right] \times [0,1]$$

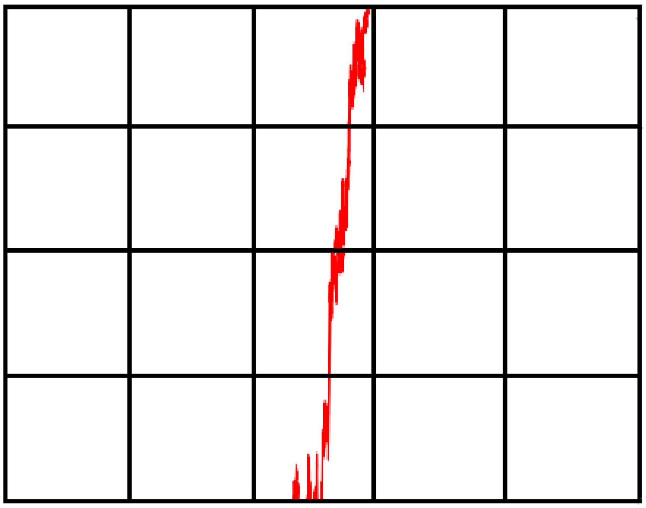


Figure 14: Example event satisfying the conditions for the lower dimension.

and $N_{n_i}^{1\times 1}((0,0), G_{X_i}^{[0,1]}) = n_i + 1$. Since the intervals I_i are disjoint, by the Borel-Cantelli Lemma, with probability one, infinitely many of the above events must happen.

Thus

$$N_{n_i}^{R_i \times R_i^{1/\alpha}}\left((a_i, X(a_i)), G_X^{I_i}\right) = n_i + 1$$

happens infinitely often and this intersection includes the whole 'middle' column. In particular these middle n_i rectangles are all in one column, so a cover by squares of the intersection of $G_X^{I_i}$ with this column will only need one column of squares. Hence, restricting to the square of side length R_i gives

$$N_{n_i}^{R_i \times R_i}\left((a_i, X(a_i)), G_X^{I_i}\right) = n_i + 1,$$

again with the 'middle' column being fully intersected. Combined with Proposition 11 this completes the proof that with probability one

$$\dim_{\mathrm{L}} G_X^{[0,1]} = 1.$$

4.3.3 Graphs of some stochastic integrals

This section is dedicated to the proof of Theorem 8. Given a function f which is zero at most only finitely often and is C^2 on some interval, say I', we can simply focus on the function restricted to I', normalising to obtain a function which is C^2 and zero only finitely often on the unit interval. We may then assume that $f(t) > 1$ for $t \in [0,1]$ by again restricting our function to an interval where the function is bounded away from zero and normalising. This is possible as the Assouad dimension is a local notion so if the dimension of some subset is full for the Assouad dimension then the whole graph will also have full dimension. Thus we assume for the rest of this proof that f is a C^2 function which is greater than 1.

Ideally we would wish to integrate by parts in the standard Riemann-Stieltjes sense

$$\int_0^t f(x)dW(x) = f(t)W(t) - \int_0^t W(x)f'(x)dx. \tag{4.1}$$

The problem is that the integral on the left side of the above equation is interpreted as the Itô integral, for which regular integration by parts does not hold. There is however a generalisation of this formula for stochastic integrals which holds as f and W are both semimartingales, see [Pro05][Chapter 2 Section 6] for further details. To be precise we should write the following equation

$$\int_0^t f(x)dW(x) = f(t)W(t) - \int_0^t W(x)f'(x)dx - [f, W]_t.$$

Here $[f, W]_t$ is the quadratic covariation between f and W which is defined as follows. Let $0 = t_1 < t_2 < \cdots < t_n = t$ be a partition $\mathcal{P}$ of $[0, t]$ and $\mathcal{P}_{\max}$ be the maximum of $t_{k+1} - t_k$ over all $k \in \{0, \dots, n-1\}$, then

$$[f, W]_t = \lim_{\mathcal{P}_{\max} \to 0} \sum_{i=1}^{n-1} (f(t_{i+1}) - f(t_i))(W(t_{i+1}) - W(t_i)).$$

The above convergence is taken in the sense of probability. By using Cauchy-Schwarz we see that:

$$[f, W]_t \leqslant [f, f]_t^{1/2} [W, W]_t^{1/2}.$$

However, it is standard that $[f, f]_t = 0$ and $[W, W]_t = t$ as f is C^1 and W is the Wiener process. So we see that the integral by parts formula (4.1) is indeed correct for this situation.

The integral

$$\int_0^t W(x)f'(x)dx$$

is defined to be a random process whose sample space is that of the Wiener process, where fixing a sample path of the Wiener process will determine the integral. We are interested in almost sure properties of this process and will do so by considering almost sure properties of the Wiener process.

The strategy for the rest of this proof is to carefully choose a path of the Wiener process whose graph has full Assouad dimension using the previous section and then note that such a path is typical. We denote the sample space of the Wiener process as Ω and for any $\omega \in \Omega$ write $W(t, \omega)$ for the realisation of the Wiener process with respect to ω.

For any fixed path ω, using the above discussion, we see that

$$\begin{aligned}
&\left|B_f\left(a_i+(k+1)\frac{R_i}{n_i},\omega\right)-B_f\left(a_i+k\frac{R_i}{n_i},\omega\right)\right| \\
&\qquad=\left|\int_{a_i+k\frac{R_i}{n_i}}^{a_i+(k+1)\frac{R_i}{n_i}} f(x)W(dx)\right| \\
&\qquad=\left|\Big[f\left(x\right)W(x,\omega)\Big]_{x=a_i+k\frac{R_i}{n_i}}^{a_i+(k+1)\frac{R_i}{n_i}}-\int_{a_i+k\frac{R_i}{n_i}}^{a_i+(k+1)\frac{R_i}{n_i}} W(x)f'(x)dx\right|, \qquad (4.2)
\end{aligned}$$

where $|\cdot|$ denotes the absolute value in this section.

We wish to bound the above. First, we see that for almost all $\omega\in\Omega$, $W(t,\omega)$ is continuous in t, and therefore there is a constant C_ω such that

$$|W(t,\omega)|\leqslant C_\omega \qquad (4.3)$$

for all $t\in[0,1]$.

The second almost sure property is described in the proof of Theorem 5, that there are infinitely many intervals $I_i=[a_i,b_i]\subset[0,1]$ of lengths R_i and a sequence $n_i\to\infty$ such that for all $k\in\{0,1,\dots,n_i-1\}$

$$\left|W\left(a_i+(k+1)\frac{R_i}{n_i},\omega\right)-W\left(a_i+k\frac{R_i}{n_i},\omega\right)\right|\geqslant R_i^{1/2}\geqslant R_i. \qquad (4.4)$$

This essentially follows from $N_{n_i}^{R_i\times R_i^{1/2}}((a_i,W(a_i),G_W^{I_i})=n_i^2$ and recalling that the Wiener process is 2-stable. Here the obtained sequence of R_i is a random subsequence of a deterministic sequence so the R_i can be controlled.

In the following discussion we shall fix a typical ω such that $W(t,\omega)$ satisfies the above two almost sure properties. In particular, we think of $C_\omega>0$ as a fixed constant.

Since f is C^2, we see that there is a constant C_f (which does not depend on i) such that for all $x\in\left[a_i+k\frac{R_i}{n_i},a_i+(k+1)\frac{R_i}{n_i}\right]$, combined with equation (4.3), we have

$$|W(x)f'(x)|\leqslant C_fC_\omega. \qquad (4.5)$$

Then using equation (4.4) the following inequality holds

$$\left|f\left(a_i+k\frac{R_i}{n_i}\right)\Big[W(x,\omega)\Big]_{x=a_i+k\frac{R_i}{n_i}}^{a_i+(k+1)\frac{R_i}{n_i}}\right|\geqslant\left|f\left(a_i+k\frac{R_i}{n_i}\right)\right|R_i^{1/2}\geqslant R_i^{1/2} \qquad (4.6)$$

and by equation (4.5)

$$\left|\int_{a_i+k\frac{R_i}{n_i}}^{a_i+(k+1)\frac{R_i}{n_i}} W(x)f'(x)dx\right| \leqslant C_f C_\omega \frac{R_i}{n_i}. \tag{4.7}$$

Since $R_i \to 0$ and $n_i \to \infty$, we can use (4.2) and Taylor's theorem for all i large enough, recalling f has a continuous second derivative, to see that

$$\begin{aligned}
&\left|B_f\left(a_i+(k+1)\frac{R_i}{n_i},\,\omega\right)-B_f\left(a_i+k\frac{R_i}{n_i},\,\omega\right)\right| \\
&\quad=\left|f\left(a_i+(k+1)\frac{R_i}{n_i}\right)W\left(a_i+(k+1)\frac{R_i}{n_i},\omega\right)\right. \\
&\qquad\qquad -f\left(a_i+k\frac{R_i}{n_i}\right)W\left(a_i+k\frac{R_i}{n_i},\omega\right) \\
&\qquad\qquad \left.-\int_{a_i+k\frac{R_i}{n_i}}^{a_i+(k+1)\frac{R_i}{n_i}} W(x)f'(x)dx\right| \\
&\quad=\left|\left[f\left(a_i+k\frac{R_i}{n_i}\right)+f'\left(a_i+k\frac{R_i}{n_i}\right)\frac{R_i}{n_i}+O\left(\left(\frac{R}{n}\right)^2\right)\right]W\left(a_i+(k+1)\frac{R_i}{n_i},\omega\right)\right. \\
&\qquad\qquad -f\left(a_i+k\frac{R_i}{n_i}\right)W\left(a_i+k\frac{R_i}{n_i},\omega\right) \\
&\qquad\qquad \left.-\int_{a_i+k\frac{R_i}{n_i}}^{a_i+(k+1)\frac{R_i}{n_i}} W(x)f'(x)dx\right| \\
&\quad=\left|f\left(a_i+k\frac{R_i}{n_i}\right)\Big[W(x,\omega)\Big]_{x=a_i+k\frac{R_i}{n_i}}^{a_i+(k+1)\frac{R_i}{n_i}}\right. \\
&\qquad\qquad +\left[f'\left(a_i+k\frac{R_i}{n_i}\right)\frac{R_i}{n_i}+O\left(\left(\frac{R}{n}\right)^2\right)\right]W\left(a_i+(k+1)\frac{R_i}{n_i},\omega\right) \\
&\qquad\qquad \left.-\int_{a_i+k\frac{R_i}{n_i}}^{a_i+(k+1)\frac{R_i}{n_i}} W(x)f'(x)dx\right|.
\end{aligned} \tag{4.8}$$

As i can be chosen to be large enough, R_i will be close to 0 and n_i will be arbitrarily large. Thus the following inequality also holds

$$R_i^{1/2} \geqslant C_f C_w \frac{R_i}{n_i} + \left|O\left(\left(\frac{R_i}{n_i}\right)^2\right)W\left(a_i+(k+1)\frac{R_i}{n_i},\omega\right) - C_f C_w \frac{R_i}{n_i}\right|.$$

Combining equations (4.5), (4.6) and (4.7) with this inequality yields

$$\begin{aligned}
&\left| f\left(a_i + k\frac{R_i}{n_i}\right)\Big[W(x,\omega)\Big]_{x=a_i+k\frac{R_i}{n_i}}^{a_i+(k+1)\frac{R_i}{n_i}} \right| \geqslant R_i^{1/2} \\
&\quad \geqslant C_f C_w \frac{R_i}{n_i} + \left| O\left(\left(\frac{R_i}{n_i}\right)^2\right) W\left(a_i + (k+1)\frac{R_i}{n_i}, \omega\right) - C_f C_w \frac{R_i}{n_i} \right| \\
&\quad \geqslant \left| f'\left(a_i + k\frac{R_i}{n_i}\right)\frac{R_i}{n_i} W\left(a_i + (k+1)\frac{R_i}{n_i}, \omega\right) \right. \\
&\qquad \left. + O\left(\left(\frac{R_i}{n_i}\right)^2\right) W\left(a_i + (k+1)\frac{R_i}{n_i}, \omega\right) - \int_{a_i+k\frac{R_i}{n_i}}^{a_i+(k+1)\frac{R_i}{n_i}} W(x)f'(x)dx \right|.
\end{aligned} \tag{4.9}$$

Applying the reverse triangle inequality to (4.8) and using the inequalities in (4.9) gives a constant C' which depends only on f and ω such that

$$\begin{aligned}
&\left| B_f\left(a_i + (k+1)\frac{R_i}{n_i}, \omega\right) - B_f\left(a_i + k\frac{R_i}{n_i}, \omega\right) \right| \\
&\quad = \left| f\left(a_i + k\frac{R_i}{n_i}\right)\Big[W(x,\omega)\Big]_{x=a_i+k\frac{R_i}{n_i}}^{a_i+(k+1)\frac{R_i}{n_i}} \right. \\
&\qquad\qquad + \left[f'\left(a_i + k\frac{R_i}{n_i}\right)\frac{R_i}{n_i} + O\left(\left(\frac{R_i}{n_i}\right)^2\right)\right] W\left(a_i + (k+1)\frac{R_i}{n_i}, \omega\right) \\
&\qquad\qquad \left. - \int_{a_i+k\frac{R_i}{n_i}}^{a_i+(k+1)\frac{R_i}{n_i}} W(x)f'(x)dx \right| \\
&\quad \geqslant \left| \left| f\left(a_i + k\frac{R_i}{n_i}\right)\Big[W(x,\omega)\Big]_{x=a_i+k\frac{R_i}{n_i}}^{a_i+(k+1)\frac{R_i}{n_i}} \right| \right. \\
&\qquad\qquad - \left| \left[f'\left(a_i + k\frac{R_i}{n_i}\right)\frac{R_i}{n_i} + O\left(\left(\frac{R_i}{n_i}\right)^2\right)\right] W\left(a_i + (k+1)\frac{R_i}{n_i}, \omega\right) \right. \\
&\qquad\qquad \left. \left. - \int_{a_i+k\frac{R_i}{n_i}}^{a_i+(k+1)\frac{R_i}{n_i}} W(x)f'(x)dx \right| \right| \\
&\quad \geqslant \left| R_i^{1/2} - C_f C_w \frac{R_i}{n_i} - \left| O\left(\left(\frac{R_i}{n_i}\right)^2\right) W\left(a_i + (k+1)\frac{R_i}{n_i}, \omega\right) - C_f C_w \frac{R_i}{n_i} \right| \right| \\
&\quad \geqslant C' R_i.
\end{aligned} \tag{4.10}$$

The above inequality holds for all $k \in \{0, \ldots, n_i - 1\}$.

Thus, as in the Lévy process setting, there exists infinitely many times where the graph will fluctuate by more than R_i within an interval of length R_i/n_i.

Moreover, following the discussion at the end of the proof of the Assouad part of Theorem 5, W has a 'zigzag' property. That is, for even integers k, we have

$$W\left(a_i + (k+1)\frac{R_i}{n_i}, \omega\right) - W\left(a_i + k\frac{R_i}{n_i}, \omega\right) > 0,$$

and for odd integers k we have

$$W\left(a_i + (k+1)\frac{R_i}{n_i}, \omega\right) - W\left(a_i + k\frac{R_i}{n_i}, \omega\right) < 0.$$

We can see that the expression inside the absolute values in (4.10) also satisfies a similar 'zigzag' property given that f is strictly positive. Therefore there is a constant $A = A(\omega, f) > 0$ such that

$$N_{n_i}^{R_i \times R_i}\left((a_i, B_f(a_i)), G_{B_f}^{I_i}\right) \geqslant A n_i^2$$

for infinitely many i. Due to the choice of ω, this concludes the proof because the above argument holds for a set of full probability $\omega \in \Omega$.

4.3.4 A remark on higher dimensional Brownian motion

The covering number results of Section 4.3.1 have natural generalisations in $\mathbb{R}^d$. Let $R_1, \ldots, R_d > 0$ be positive numbers and $n \in \mathbb{N}$. Given a point $\mathbf{a} \in \mathbb{R}^d$, we define $W_n^{R_1 \times \cdots \times R_d}(\mathbf{a})$ as the following collection of sets:

$$\Bigg\{ D_{i_1,\ldots,i_d} + \mathbf{a} \,|\, D_{i_1,\ldots,i_d} = \left[\frac{i_1}{n}R_1, \frac{i_1+1}{n}R_1\right] \times \cdots \times \left[\frac{i_d}{n}R_d, \frac{i_d+1}{n}R_d\right], \\ i_j \in \{0, \ldots, n-1\}, j \in \{1, \ldots, d\} \Bigg\}.$$

We see that $W_n^{R_1 \times \cdots \times R_d}(\mathbf{a})$ is a collection of rectangles with disjoint interiors. The graph of a function f which intersects a full collection of $W_n^{R_1 \times \cdots \times R_d}(\mathbf{a})$ infinitely many times will have full Assouad dimension. This follows from the definitions of dimensions.

Using a similar argument as the one in Section 4.3.2, it can be shown that this full intersection of the graph of a function and $W_n^{R_1 \times \cdots \times R_d}(\mathbf{a})$ can occur with positive probability and again by Borel-Cantelli, we obtain infinitely many occurrences almost surely. Hence the Assouad dimension of d-dimensional Brownian motion is d.

4.4 Pushforwards of measures onto graphs of Brownian motion

We now turn our attention to pushforwards of measures onto graphs of Lévy processes.

Recall the definition of the graph of a Lévy process X restricted to the unit interval

$$G_X^{[0,1]} = \{(t, X(t)) \colon t \in [0,1]\} .$$

There is a naturally associated function $f : [0,1] \to \mathbb{R}^2$ which maps the unit interval to the graph of the process, that is $f \colon t \mapsto (t, X(t))$. We bring attention to this function now as it will be the map we wish to use to construct pushforward measures, and for the rest of this chapter f should be assumed to be this map.

This leads us to the question of this section: given a doubling measure μ on the unit interval, is $f_*\mu$ also doubling? A similar question can be posed for uniformly perfect measures. So far we have shown the Assouad dimension of $G_X^{[0,1]}$ is almost surely 2, see Theorem 5, so there must exist at least one doubling measure on the graph. However, most measures on the graph might not even be doubling. For the Hausdorff dimension, the proof by Taylor shows that the Hausdorff dimension of the pushforward of Lebesgue measure almost surely attains the dimension of the graph itself. It turns out that this is usually not the case for the regularity dimensions.

Theorem 15 ([How19])**.** *Let μ be a doubling measure on $[0,1]$ and X a stable Lévy process with the distribution $X(1)$ being non trivial on $\mathbb{R}$. Then $f_*\mu$ is almost surely not doubling on $G_X^{[0,1]}$. Also, $f_*\mu$ is almost surely not uniformly perfect.*

Trivially this implies the upper regularity dimension of $f_*\mu$ is almost surely infinity and the lower regularity dimension is almost surely zero. Therefore any measure whose upper regularity dimension approximates the dimension of the graph is highly dependent on the specific graph and so there is no one measure that attains the dimension for typical realisations, unlike the Hausdorff case.

The following proof of this theorem is inspired by the ideas used to prove Theorem 5 however an extra case must be considered for when $\alpha < 1$. It is perhaps interesting to note we obtain a result for $\alpha < 1$ in the measure theoretic case but not the set analogue. This is simply due to the techniques used in the proofs, this second technique will be more robust to $\alpha < 1$ causing rectangles to be wide rather than tall. This result, in a way, does indicate that the Assouad dimension of the graph of a Lévy process with $\alpha < 1$ should also be full.

Proof of Theorem 15. Choose a Lévy process X satisfying the conditions in Theorem 15 which is α-stable and fix the graph $G_X^{[0,1]}$ realised by this process. Start by assuming $\alpha \geqslant 1$, the proof will work in the same way for $\alpha < 1$ given a slight modification which will be commented on later in the proof. A measure μ is taken to be a doubling measure on the unit interval. Recall f is defined to be the function which maps the unit interval to the graph of our Lévy process and $f_*\mu$ is the pushforward measure of μ onto the graph that we wish to study.

We start by calculating the almost sure upper regularity dimension of $f_*\mu$. Let $s > 0$. The general strategy for this proof is to find a sequence of events that are all independent and have positive probability. Then a simple application of the Borel-Cantelli lemma will yield that almost surely these events will happen infinitely often. By choosing our events carefully this will yield a sequence of balls that show the upper regularity dimension of the pushforward measure must be greater than s. As s is arbitrary, this will conclude the proof.

Given our α-scaling Lévy process, we denote the rectangle centered at $a \in [0,1]$ with side lengths R_1, R_2 by $Rec(a, R_1, R_2) = I(a, R_1) \times I(X(a), R_2)$ where $I(b, R) = [b - R/2, b + R/2]$ is just an interval of length R and centre b.

The particular events E_i we are interested in are defined as follows: let $x_i \in [0,1]$, $R_i > r_i > 0$ and $\beta > 1$, then E_i is the intersection of the events $G_X^{I(x_i, R_i^\alpha)} \subset Rec(x_i, R_i^\alpha, R_i)$ and $Rec(x_i, r_i, r_i^{1/\alpha}) \cap G_X^{I(x_i, r_i)} = G_X^{I(x_i, r_i^\beta)}$. These events are chosen so that the measure on the graph will be 'large' on the rectangle of small side length R^α but 'small' on the rectangle of small side length r. Figure 16 is a geometric representation of such an event.

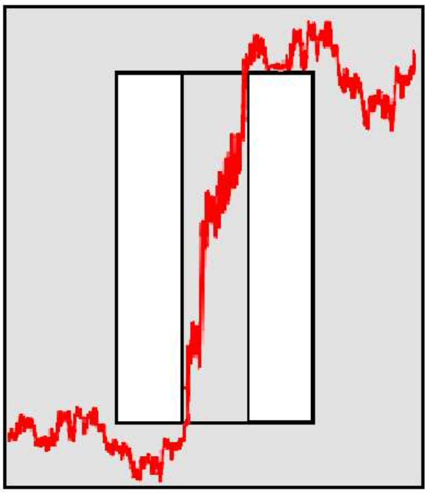


Figure 16: Example of an event E_i, the grey areas are where the graph intersects the rectangles whilst the graph will not intersect the white areas. An example of a graph satisfying this event is in red.

Given any sequences $x_i \in [0,1]$, $R_i > r_i > 0$ and $\beta > 1$ we can consider the associated events E_i as above. To make sure the 'smaller' rectangle is actually smaller, assume $R_i^\alpha > r_i$ without loss of generality. If $Rec(x_m, R_m^\alpha, R_m) \cap Rec(x_n, R_n^\alpha, R_n) = \emptyset$ for all $m \neq n$, then the events are all independent due to the independent increment property of the Lévy process. As long as the distribution of $X(1)$ is non-vanishing on the unit interval, the probability of any of these events is positive.

We can now choose our sequence of events. Start by picking any disjoint and strictly increasing sequence of reals x_i, $\{1 - 2^{-i}\}_{\mathbb{N}}$ would suffice. Then the R_i are taken so that the intervals $I(x_i, R_i)$ do not overlap ensuring independence, say 4^{-i}. Initially any sequence of r_i can be chosen as long as $R_i/r_i \to \infty$ and, again, $R_i^\alpha > r_i$ for each i. The sequence of β will be fixed later, for now it is just a real greater than 1. As the process is α-stable one can map $Rec(x_i, R_i^\alpha, R_i)$ onto the unit square via an affine map T and the image of the graph under this transformation, denoted $G_{X_i}^{[0,1]}$, will have distribution X_i equal to the original distribution $X(t)$ as it is scaled following the definition of α-scaling, so $X(t) = X(R_i^\alpha t)/R_i = X_i(t)$ in distribution. This is the same process as in the proof of Theorem 5. Therefore the probability of an event E_i is equal to the probability the graph of X_i stays in the unit square and

$$Rec(1/2, r_i/R_i^\alpha, r_i^{1/\alpha}/R_i) \cap G_{X_i}^{I(1/2, r_i/R_i^\alpha)} = G_{X_i}^{I(1/2, r_i^\beta/R_i^\alpha)}.$$

Thus the probability of E_i depends solely on the ratio $R_i/r_i = q_i$. If $\sum P(E_i)$ diverges then the conditions for Borel-Cantelli are satisfied and the argument continues. However, if not, the sequence r_i is modified in the following way. Each i gives us a ratio q_i and a probability $P(E_i)$. Construct a function $g\colon \mathbb{N} \to \mathbb{N}$ such that $g(n) = \lceil \frac{1}{nP(E_n)} \rceil$ for all $n \in \mathbb{N}$. Then, keeping R_i fixed, change the r_i so that each ratio q_i is repeated $g(i)$ many times. For instance, if $g(1) = 3$ then r_1, r_2 and r_3 are chosen so that $R_1/r_1, R_2/r_2$ and R_3/r_3 all give the same $P(E_1)$ and r_4 then is chosen with respect to $P(E_2)$ etc. The new sequence is constructed such that $\sum P(E_i)$ diverges, satisfying the conditions for the Borel-Cantelli Lemma.

Hence, by the Borel-Cantelli Lemma, infinitely many E_i occur with probability one. So there are sequences $x_i \in [0,1]$, $R_i > r_i > 0$ and $\beta > 1$ such that, with full probability, all of the events E_i happen and $R_i/r_i \to \infty$.

Given a specific event E_i we wish to consider the measure of the rectangles. The ratio of measures of such rectangles is determined by the original measure on the interval. We let $t = \underline{\dim}_{\mathrm{reg}}\mu/2$, this is just to have a number for which the following bound holds but is also fixed and positive due to Proposition 4 from the previous chapter. Thus we obtain the following bound:

$$\frac{f_*\mu(Rec(x_i, R_i^\alpha, R_i))}{f_*\mu(Rec(x_i, r_i, r_i^{1/\alpha}))} = \frac{\mu(B(x_i, R_i^\alpha))}{\mu(B(x_i, r_i^\beta))} \geqslant C \left(\frac{R_i^\alpha}{r_i^\beta}\right)^t,$$

where C comes from the definition of the lower regularity dimension.

The only variable left to be fixed is β. We wish to have the above ratio greater than $C(R_i/r_i)^s$. After a short calculation, it is clear that this is always true if $\beta \geqslant \alpha + s/t$. Thus by choosing such a β we have

$$\frac{f_*\mu(Rec(x_i, R_i^\alpha, R_i))}{f_*\mu(Rec(x_i, r_i, r_i^{1/\alpha}))} \geqslant C \left(\frac{R_i^{\alpha t}}{r_i^{\alpha t+s}}\right) \geqslant C \left(\frac{R_i^{\alpha t+s}}{r_i^{\alpha t+s}}\right) \geqslant C \left(\frac{R_i}{r_i}\right)^s.$$

To show the upper regularity dimension is greater than s we need to consider balls not rectangles. Thankfully due to our construction $B(x_i, R_i) \supset Rec(x_i, R_i^\alpha, R_i)$ and $B(x_i, r_i) \subseteq Rec(x_i, r_i, r_i^{1/\alpha})$. Hence

$$\frac{f_*\mu(B(x_i, R_i))}{f_*\mu(B(x_i, r_i))} \geqslant \frac{f_*\mu(Rec(x_i, R_i^\alpha, R_i))}{f_*\mu(Rec(x_i, r_i, r_i^{1/\alpha}))} \geqslant C \left(\frac{R_i}{r_i}\right)^s,$$

completing the proof.

When $\alpha < 1$ much of the above argument still holds, that is we can construct the same events E_i and due to the distribution of $X(1)$ they will have positive probability. Since the process is stable we can choose an infinite sequence of events such that the sum of the probabilities is infinite and then Borel-Cantelli says that infinitely many must occur with probability one. Note however that the rectangle $Rec(x_i, R_i^\alpha, R_i)$, for suitable variables, is now wider than it is tall. Therefore the final step where we switch back from rectangles to balls needs to be modified and a different β must be chosen. We assume the same set-up has been followed as the $\alpha \geqslant 1$ case and arrive at

$$\frac{f_*\mu(Rec(x_i, R_i^\alpha, R_i))}{f_*\mu(Rec(x_i, r_i, r_i^{1/\alpha}))} \geqslant C \left(\frac{R_i^\alpha}{r_i^\beta}\right)^t.$$

The natural upper bound for this with respect to balls is

$$\frac{f_*\mu(B(x_i, R_i^\alpha))}{f_*\mu(B(x_i, r_i^{1/\alpha}))} \geqslant \frac{f_*\mu(Rec(x_i, R_i^\alpha, R_i))}{f_*\mu(Rec(x_i, r_i, r_i^{1/\alpha}))}$$

where R_i and r_i must be chosen so that $r_i^{1/\alpha} \leqslant R_i^\alpha$ which follows from the previous conditions. We wish to show that

$$\frac{f_*\mu(B(x_i, R_i^\alpha))}{f_*\mu(B(x_i, r_i^{1/\alpha}))} \geqslant C \left(\frac{R_i^\alpha}{r_i^{1/\alpha}}\right)^s$$

and by choosing $\beta \geqslant s/(t\alpha)$, as long as $s \geqslant t$, it follows that

$$\frac{f_*\mu(B(x_i, R_i^\alpha))}{f_*\mu(B(x_i, r_i^{1/\alpha}))} \geqslant C\left(\frac{R_i^\alpha}{r_i^\beta}\right)^t \geqslant C\left(\frac{R_i^\alpha}{r_i^{1/\alpha}}\right)^s$$

as desired.

For the lower regularity dimension it suffices to change the events E_i in the following way. Assuming $\alpha > 1$, let $x_i \in [0,1]$, $R_i > r_i > 0$ and $\beta < 1$, then E_i is the event where $G_X^{I(x_i,R_i)} \cap Rec(x_i, R_i, R_i^{1/\alpha}) \subseteq Rec(x_i, r_i^\beta, R_i^{1/\alpha})$ and $G_X^{I(x_i,r_i)} \subseteq Rec(x_i, r_i^\alpha, r_i)$. The previous argument then works in much the same way, showing that the lower regularity dimension of $f_*\mu$ is zero as desired.

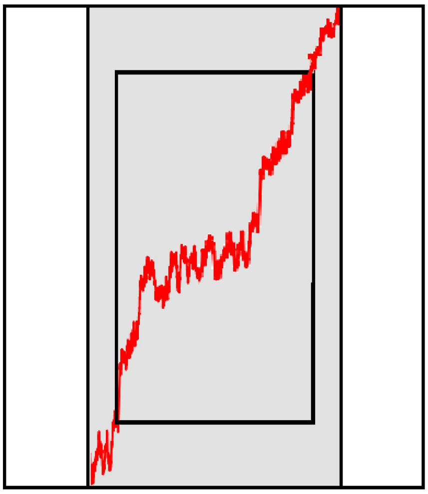


Figure 17: Example of an event E_i for the lower regularity dimension, the grey areas are where the graph intersects the rectangles whilst the graph will not intersect the white areas.

$\square$

Chapter 5

Open Problems

To conclude this thesis, we will recall and expand upon some of the questions that have been raised so far. This will merely be a list of problems that we found interesting when writing this work and is in no way exhaustive, but should offer a number of options for future work. In this chapter we will follow the general order in which the work was presented in this thesis without recalling specific notation that was introduced previously.

5.1 Weak tangent measures

In Chapter 2 we introduced the concept of weak tangent measures and Theorem 7 showed that the upper regularity dimension of a measure is bounded below by the upper regularity dimension of its weak tangent measures. This is a natural analogue of the set theoretic setting, where weak tangent sets have Assouad dimension less than or equal to the Assouad dimension of the original set. Recently, in [FHKY19], it was shown that for any $\mathcal{F}_\sigma$ set $\Delta \subseteq [0,1]$ which contains its infimum and supremum, there exists a set $F \subseteq [0,1]$ such that the set of all possible Hausdorff dimensions of the weak tangents of F equals Δ. It is natural to ask if the same holds in the measure theoretic setting.

Question 1. *What set theoretic restrictions are there on the set $\Delta \subseteq (0,\infty)$ which contains its infimum and supremum such that one can find a locally finite, Borel measure μ supported on a subset of $\mathbb{R}$ so that*

$$\{\underline{\dim}_{\mathrm{H}}\hat{\mu} \colon \hat{\mu} \text{ is a weak tangent measure of } \mu\} = \Delta?$$

Note in the above question it is assumed that the sets Δ contain their infimum and supremum, this is due to the fact that the Assouad and lower dimensions of a set is always attained by their microsets. It is perhaps reasonable to assume that this also holds for measures but this should be investigated further.

Question 2. *For any measure μ supported on $\mathbb{R}$, is it true that there exists a weak tangent measure $\hat{\mu}$ such that*

$$\overline{\dim}_{\mathrm{reg}}\mu = \overline{\dim}_{\mathrm{reg}}\hat{\mu}$$

and another weak tangent measure $\hat{\mu}'$ such that

$$\underline{\dim}_{\mathrm{reg}}\mu = \underline{\dim}_{\mathrm{reg}}\hat{\mu}'?$$

5.2 Self-similar and self-affine measures

There has been much work calculating the dimensions of self-similar and self-affine sets in a range of different settings. Here we restricted to self-similar measures on sets satisfying the strong separation condition. This is required to ensure these measures are doubling. However, there are still many examples of doubling self-similar measures for which only a weaker separation condition holds, such as the open set condition. Understanding which measures are still doubling when the separation conditions are weakened would be the first step in calculating the regularity dimensions of these measures.

Another strand of research can be found in [HHT18], where the quasi-Assouad dimension of self-similar measures was studied. This different notion of dimension only depends on the measure being quasi-doubling, a weaker property than doubling and so only a weaker separation is required for these calculations. The quasi-Assouad dimension is closely related to the upper regularity dimension so it is likely that the ideas in [HHT18] could be used to further our understanding of the regularity dimensions too.

For self-similar measures there are a number of different separation conditions with various notions of dimensions and regularity to consider so, for brevity, we simply state this as the following question which can also be found on page 30.

Question 3. *What are the regularity dimensions of self-similar measures on sets not satisfying the strong separation condition?*

After studying self-similar measures it was natural to turn our attention to self-affine measures on Bedford-McMullen sponges. The regularity dimensions were calculated for such measures assuming two conditions - the very strong separation condition and inequality of the contraction ratios. The first was needed to ensure the measures were doubling; as before it would be interesting to ask which, if any, conditions guarantee doubling. The second condition is important in the proof of our theorem and was first remarked upon in [FH15] in the set theoretic setting. This was removed for the Assouad and lower dimensions of sets in [How16] and it seems reasonable that the techniques in that paper can be used in a similar way for the regularity dimensions of measures.

As noted in Section 2.2.4, there are a number of generalisations of Bedford-McMullen sponges that have been considered in the literature, covering several different notions of dimension. Finding the regularity dimensions of measures in the examples for which the Assouad and lower dimensions have been studied should be possible by combining the measure theoretic techniques used here with the new tools introduced in the set theoretic settings. For the remaining cases, it would be worth turning our attention to the study of the Assouad dimension first, before attempting to find the regularity dimensions. All these problems were resumed on page 35 in the following question.

Question 4. *What can be said about the regularity dimensions of self-affine measures on sets which do not satisfy the VSSC? What are the regularity dimensions of self-affine measures on more general carpets, such as Lalley-Gatzouras carpets?*

5.3 Measures on sequences

Sequences have proven to be particularly interesting objects of study with respect to the Assouad dimension in terms of how different the results often are compared to the more common dimensions. The regularity dimensions of measures on sequences have also shown to exhibit similar phenomena and it would be helpful to have further examples at hand to better understand these dimensions. A natural way of obtaining more examples would be to consider different decay rates, such as stretched exponential, to see if the measures remain doubling as long as the decay rates of the sequence and the measure are comparable.

One could also look at more general sequences, for instance by removing the decreasing gap condition. However, it would be important to maintain some reasonable conditions to ensure the set and measure continue to resemble genuine sequences. How to formalise this idea at the moment is not clear, further study would have to be undertaken to understand when these sequences develop behaviour that we do not consider 'normal' for such objects. As this second part is less precise we omit formally stating it as a question for now, simply recalling the problem posed on page 37.

Question 5. *How does the upper regularity dimension of measures defined on sequences behave for different decay rates, such as stretched exponential decay?*

5.4 Quantifying doubling and uniform perfectness

In Chapter 3 it was seen that there is a link between the upper regularity dimension and the doubling constants, similarly for the lower analogues. However, this only relies on the limiting behaviour of the doubling constants $C(\theta)$ as θ tends to infinity and so not does not imply any obvious behaviour for small θ. The following is a natural question to then ask.

Question 6. *Given $a, b, \theta > 1$, does there exists a measure μ such that $a = C(\theta)$ and $b = K(\theta)$?*

There are a number of extensions to this question that can be posed, such as replacing a, b and θ by sequences a_i, b_i, θ_i and asking one measure to have constants $C(\theta_i) = a_i$ and $K(\theta_i) = b_i$ for all i. Or one could keep the question as is but simultaneously impose specific regularity dimensions on the measure.

Whilst these questions are interesting, it is currently difficult to calculate these optimal constants even in the simple self-similar setting for specific θ. Developing a better understanding of how these constants behave would be the first step in finding an answer. Doing this would also improve our study of the relations between the regularity dimensions and doubling and uniform perfectness. For instance, in Proposition 4 of Chapter 3, we showed that a doubling measure on a uniformly perfect space has not only positive lower regularity dimension, but also a lower bound to this dimension can be stated explicitly in terms of the doubling constant for a specific θ. If a relation between the doubling constants for various values of θ can be found, then it is plausible that this bound can be improved.

To start this analysis we state the following problem which should be achievable.

Question 7. *Given a self-similar set satisfying the OSC and a self-similar measure on this set, what are the optimal constants $C(\theta)$ and $K(\theta)$ for all $\theta > 1$?*

5.5 Quantifying doubling and uniform perfectness

To finish Chapter 3 we showed that the lower regularity dimension can have applications in Diophantine approximation, providing a hopefully more applicable version of a Theorem in [BGSV16] as the regularity dimensions have now been calculated for a range of different examples. There have been other studies related to the work of [BGSV16] which rely on different regularity properties of measures and it seems reasonable to ask if and how the upper and lower regularity dimensions interact with these notions. A common condition that occurs in this setting is that the measure be 'locally doubling'. This lends credibility to the idea that the upper regularity dimension is also involved in some way. We leave this investigation open by recalling the question stated on page 64.

Question 8. *Is there a relation between the regularity dimensions and other notions of regularity of measures used in Diophantine approximation?*

5.6 Graphs of Brownian motion

There are several cases left unresolved in the study of the dimensions of graphs of Brownian motion and measures supported on them. Regarding the Assouad dimension of the graph, it is still not clear what the dimension will be when the α-stable Lévy process has $\alpha < 1$. Following the usual behaviour of the Assouad dimension, it is likely almost surely full but this is not clear in this setting. Any proof would likely need new ideas compared to the ones used in this thesis for the case $\alpha > 1$. The following is a slightly reworded version of the question from page 76.

Question 9. *For any $\alpha \in (0, 1)$, what is the Assouad dimension of the graph of an α-stable Lévy process?*

It would also be natural to consider the lower dimension of the graphs of random process which are functions defined as stochastic integrals. The proof for the Assouad dimension analogue relied heavily on understanding the behaviour of the Wiener process as seen in the calculation of the dimension of the graph of the Wiener process. A similar idea was used for the lower dimension of graphs of Lévy processes, so it should be possible to combine these ideas and the lower dimension of the graph of stochastic integrals is likely almost surely 1. However, one must be careful in the proof as many of the bounds used for the Assouad dimension do not necessarily hold in the other direction. The question on what the lower dimension actually is was stated on page 77.

Question 10. *Can a lower dimension analogue of Theorem 8 (from Chapter 3) be found? Given that this result mirrors the Assouad dimension of the graphs of Brownian motion it is natural to conjecture that the lower dimension in this setting will be almost surely 1.*

List of notation

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Appendix

Many of the pictures used in this thesis were created in Mathematica or Python using a simple chaos game algorithm. The below algorithms can be run directly in Mathematica to obtain similar images. Similar Python code is available at https://github.com/douglashowroyd/IFSGen.

The following code can be easily modified to change the output. If a higher quality image is desired then the integer in the first line (currently 10000) should be increased, note that this will proportionally increase the run time and the program might not work if made too high.

The sections contained within the square brackets following a *Which* define the functions in the IFS. In the higher dimensional versions there is a *Which* for each dimension and each one contains the function's action on a specific coordinate. One can also change the number of functions in the IFS by simply changing the 3 in the first line (4 in the 3-dimensional code) to the desired number of maps and adding the functions to the corresponding *Which* line, for example, in the format ‘ j==4, x/5+1/5 ’ .

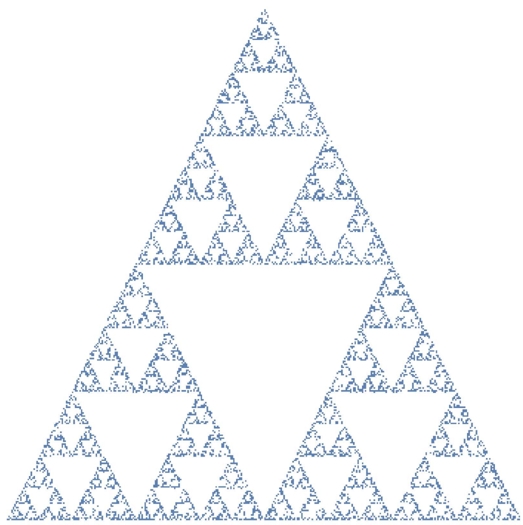
Mathematica code for 1-dimensional images.

```
functions=RandomInteger[{1,3},10000];

f[j_,x_] := Which[ j ==1, x/5, j==2, x/5+2/5, j==3, x/5+4/5 ];

sample={0};

For [ i=1, i<=Length[functions], i++,
AppendTo[ sample, { f[functions[[i]],sample[[i ]], 1 }]
];

ListPlot[ sample, PlotStyle -> PointSize[0.000000001], Axes -> {
    False,False}]
```


The image generated by the above code, here a self-similar set in ambient spatial dimension 1.

Mathematica code for 2-dimensional images.

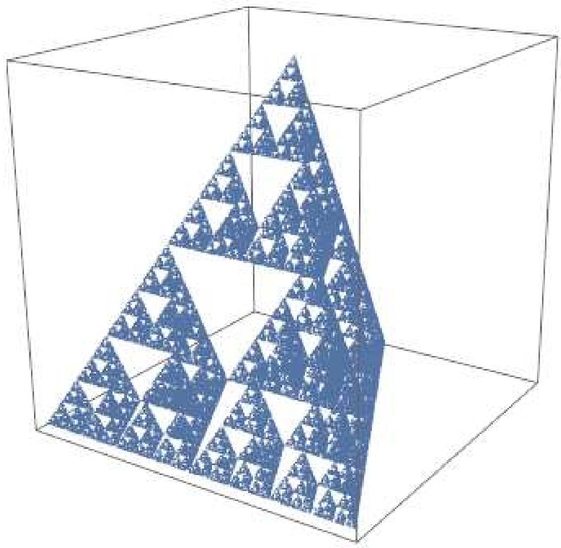
```
functions=RandomInteger[{1,3},10000];

f1[j_,x_] := Which[ j ==1, x/2, j==2, x/2, j==3, x/2+1/2 ];

f2[j_,y_] := Which[ j ==1, y/2, j==2, y/2+1/2, j==3, y/2+1/4 ];

sample={{0,0}};

For [ i=1, i<=Length[functions], i++,
AppendTo[ sample, { f2[functions[[i]],sample[[i ]][[1]]], f1[functions[[i
    ]], sample[[i ]][[2]]] }
];

ListPlot[ sample, PlotStyle -> PointSize[0.000000001], AspectRatio
    -> Automatic, Axes -> {False,False}]
```



The image generated by the above code, here a Sierpiński triangle.

Mathematica code for 3-dimensional images.

```
functions=RandomInteger[{1,4},10000];

f1[j_,x_] :=Which[ j ==1, x/2,
j==2, x/2,
j==3, x/2+1/2,
j==4, x/2+1/4
];

f2[j_,y_] := Which[ j ==1, y/2,
j==2, y/2+1/2,
j==3, y/2+1/4,
j==4, y/2+1/4
];

f3[j_,z_] := Which[ j ==1, z/2,
j==2, z/2,
j==3, z/2 ,
j==4, z/2+1/2
];

sample={{0,0,0}};

For [ i=1, i<=Length[functions], i++,
AppendTo[ sample, {
f1[functions [[ i ]], sample[[i  ]][[1]]],
f3[functions [[ i ]], sample[[i  ]][[2]]],
f2[functions [[ i ]], sample[[i  ]][[3]]]
}]
];

ListPointPlot3D[ sample, PlotStyle -> PointSize[0.000000001],
    BoxRatios -> Automatic, Ticks -> None]
```



The image generated by the above code, here a Sierpiński tetrahedron.